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Exploring A New Class of Inequality Measures and Associated Value Judgements: Gini and Fibonacci-Type Sequences*

John Creedy† and S. Subramanian‡

Abstract

This paper explores a single-parameter generalization of the Gini inequality measure. Taking the starting point to be the Borda-type social welfare function, which is known to generate the standard Gini measure, in which incomes (in ascending order) are weighted by their inverse rank, the generalisation uses a class of non-linear functions. These are based on the so-called 'metallic sequences' of number theory, of which the Fibonacci sequence is the best-known. The value judgements implicit in the measures are explored in detail. Comparisons with other well-known Gini measures, along with the Atkinson measure, are made. These are examined within the context of the famous 'leaky bucket' thought experiment, which concerns the maximum leak that a judge is prepared to tolerate, when making an income transfer from a richer to a poorer person. Inequality aversion is thus viewed in terms of being an increasing function of the leakage that is regarded as acceptable.

JEL Classification: D30, D31, D63

Keywords: Income inequality, Gini coefficient, extensions of Gini, social welfare functions, equally distributed equivalent income, Atkinson, inequality aversion, value judgements, efficiency and equity, leaky bucket experiment

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1 Introduction

The subject of inequality measurement underwent a 'revolution' around fifty years ago, stimulated by the seminal contributions of Kolm (1969) and Atkinson (1970). The key feature of subsequent research is that the analysis of inequality is carried out within the broad context of 'welfare economics', instead of being treated purely as a statistical phenomenon.[1] Analysis is explicitly linked to an evaluation function, referred to as a 'social welfare function', that represents the views regarding total income and its distribution, of an independent judge. Despite the term 'social', the views represent the value judgements of a hypothetical independent individual, and are not meant to represent a society or other aggregate. The approach allows for the presentation of results – for example, comparisons between distributions – that are implied by different values. Rather than imposing an analyst's value judgement, results for different clearly-specified values allow readers to form their own views. This naturally raises the important question of how an evaluation function can be specified in a way that makes views explicit, sufficiently precise, and worthy of consideration in that it may be thought to represent the views of a sufficiently large number of judges.

The aim of this paper is to explore the value judgements involved in a new class of inequality measures, introduced by Subramanian (2021), which involve modifications of the Gini measure. In particular, the weights attached to individual incomes, which depend on the individual's position or rank order in the distribution, are specified using a class of non-linear functions of the rank order. This is shown to contrast with the standard Gini measure, for which the weights are obtained as a simple linear function (the inverse rank) of the order obtained when individuals are ranked in ascending order of their incomes.

The specific approach to generalizing Gini considered in this paper resides in seeing the Social Welfare Function associated with the Gini index – involving inverse ranks – as being a special case of a class of welfare functions which 'mirror' what are known as 'metallic sequences' of number theory. The most famous member of this class is the Fibonacci sequence, which appears in many unexpected aspects of natural phenomena and human endeavour – in botany, biology, music, art, and architecture.

Section 2 provides some background by rehearsing the basic approach to inequality measurement via social welfare functions and inequality measures defined in terms of the proportional deviation of the arithmetic mean income from an 'equally distributed equivalent' income. The Atkinson inequality measure, along with the standard Gini

[1] Indeed, a clear dichotomy can no longer be drawn between descriptive/statistical measures and so-called 'ethical' measures, since it is usually possible to identify implicit value judgements associated with earlier inequality measures; see, for example, Shorrocks (1988).

and various extensions, are discussed. This provides the basis for the new class of Gini-type measures, presented in Section 3. In order to make the value judgements, implicit in the measures, as transparent as possible, these are explored in Section 4. This section also provides comparisons with other well-known Gini measures, along with the Atkinson measure. The value judgements are examined within the context of the famous 'leaky bucket' thought experiment, which concerns the maximum leak that a judge is prepared to tolerate, when making an income transfer from a richer to a poorer person. Inequality aversion is thus viewed in terms of being an increasing function of the leakage that is regarded as acceptable. Conclusions are in Section 5.

2 Inequality and Welfare

This section provides the context and background for the later discussion. The extensive literature on inequality measures is often highly technical, but the aim here is to introduce the basic concepts only and lay the groundwork for the discussion of a new class of measures.[2] Subsection 2.1 briefly summarises Atkinson's (1970) influential contribution. The standard Gini, along with a class of one-parameter 'extended Ginis', is examined in subsection 2.2. Subsection 2.3 shows how the basic approach of Atkinson, with a different form of social welfare function, can lead to a Gini measure.

2.1 Atkinson's Insights

The basis of the approach taken in Atkinson's famous paper, is that he considered a class of welfare, or evaluation, functions of the form:

$$W = W\left(x_1, x_2, ...x_n\right) \tag{1}$$

where x_i is individual i's income, for $i = 1, ..., n$. The distribution is thus evaluated using a 'social welfare function' expressed in terms of the incomes of all individuals, who are considered to have no relevant non-income differences. Atkinson explored functions of the form:

$$W = \sum_{i=1}^{n} \Phi\left(x_i\right) \tag{2}$$

Here the function $\Phi\left(x_i\right)$ represents the contribution of income, x_i, to the overall evaluation. Importantly, it does not represent person i's utility function. Basic requirements are that $d\Phi\left(x_i\right)/dx_i \geq 0$, and $\Phi\left(x_i\right)$ is concave, so that $d^2\Phi\left(x_i\right)/dx_i^2 \leq 0$. The latter

[2] Earlier papers on extensions to Gini include, for example, Donaldson and Weymark (1980), Kakwani (1980), Yitzhaki (1983), Chakravarty (1988), Shorrocks and Slottje (2002), and Chameni (2006, 2008).

property implies that the value judgments conform with the Principle of Transfers, according to which a rank-preserving income transfer from 'richer to poorer' increases W, and thus represents an improvement.[3]

The recognition that an income transfer, taking pace anywhere in the distribution, that satisfies the Principle of Transfers must move the Lorenz curve closer to the diagonal line of equality, led Atkinson to the statement that, for distributions with the same arithmetic mean income, a Lorenz curve that 'dominates' another (lies everywhere closer to the line of equality) must imply a higher value of W, for any evaluation function of the form in (2). However, in measuring the precise degree of inequality, Atkinson did not refer to the Lorenz curve (previously the inspiration for Gini's measure), but first defined an 'equally distributed equivalent' income, x_E, defined by:

$$W\left(x_E, x_E, ...x_E\right) = W\left(x_1, x_2, ...x_n\right) \tag{3}$$

Hence, x_E is that income which, if obtained by each individual, gives the same value of W as the actual distribution. The next step was to define a class of inequality measures given by:

$$I_A = 1 - \frac{x_E}{\bar{x}} \tag{4}$$

That is, I_A is the proportional difference between the equally distributed equivalent income and the arithmetic mean, $\bar{x}$. Atkinson's measure can thus be calculated for any $\Phi\left(x_i\right)$ satisfying the basic properties mentioned above, and which can be inverted, allowing (3) to be solved. Taking a lead from the then-recent work on risk aversion, Atkinson confined his attention to $\Phi\left(x_i\right)$ of the form, for $\varepsilon \neq 1$:

$$\Phi\left(x_i\right) = \frac{x_i^{1-\varepsilon}}{1-\varepsilon} \tag{5}$$

and for $\varepsilon = 1$, $\Phi\left(x_i\right) = \log\left(x_i\right)$. This function thus represents the views of an independent judge who has a constant relative aversion to inequality, of $\varepsilon = x_i \Phi''\left(x_i\right)/\Phi'\left(x_i\right)$.[4] It is easily seen that x_E can be solved as:

$$x_E = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{1-\varepsilon}\right]^{1/(1-\varepsilon)} \tag{6}$$

[3] Other typical features of the type of social welfare function advanced by Atkinson include the properties of symmetry (welfare outcomes are independent of the precise personal identities of individuals), continuity (welfare does not change abruptly for small changes in individual welfare levels), and population-neutrality (the welfare function is invariant with respect to population replications).

[4] Constant absolute aversion can easily be introduced by instead writing $\Phi\left(x_i\right) = 1 - \exp\left(\beta x_i\right)$, where β is constant absolute aversion, $-\Phi''\left(x_i\right)/\Phi'\left(x_i\right)$. In addition, an intermediate case is given by $\Phi\left(x_i\right) = \left(x_i + x_0\right)^{1-\varepsilon}/\left(1-\varepsilon\right)$ for $\varepsilon \neq 1$. However, these alternatives have received little attention compared with the constant relative inequality aversion case used by Atkinson.

from which I_A is directly obtained from (4). It is therefore a simple matter to examine the implications of adopting any specified degree of relative inequality aversion, and to show that the rankings of a range of distributions by inequality can change substantially as ε is varied.

In helping users of data to obtain a clear idea of the implications of different ε values, Atkinson was aware that its effect is not immediately obvious. To provide a helpful explanation of the 'meaning' of different values of ε, he suggested using the 'leaky bucket' mental experiment. This amounts to asking, for any specified transfer, what a judge is prepared to loose in the transfer process and still support it. This amounts, for small transfers, to considering the slope, at the relevant point, of associated 'social indifference curves', which show, for example, combinations of x_j and x_k for which the judge is indifferent. In the case of constant relative inequality aversion, such curves are known to have the same slope along any ray from the origin.[5] The leaky bucket experiment is discussed further below.

2.2 The Gini Measure

Atkinson's discussion of the Lorenz curve was solely in the context of welfare dominance results. But of course it is well known that the Gini measure, G, first defined as half the ratio of the average absolute difference between all pairs of incomes to the arithmetic mean (or half the relative mean difference), can be obtained as an area measure of the 'distance' between the Lorenz curve and the line of equality. A number of different expressions can be found for this area. Many different formulae are available for the Gini, but one of the most frequently used, for ungrouped data, is the following, obtained using the standard trapezoidal rule:

$$G = \frac{n+1}{n} - \frac{2}{n^2 \bar{x}} \sum_{i=1}^{n} (n+1-i)\, x_i \tag{7}$$

where incomes are arranged in non-descending order such that $x_1 \leqslant x_2 \leqslant .... \leqslant x_n$. Atkinson's work stimulated a search for the clarification of value judgements that would give rise to a Gini measure. Given the origin of the Gini in terms of a relative mean-difference measure, Sen (1973) showed that it can be derived from a welfare function involving all pairwise comparisons, where welfare attached to any pair is the minimum income of the two.

Subsequently, it was found that a covariance form of the Gini measure, where $F(x)$

[5] The experiment has formed the basis of attempts to measure indirectly the nature of individuals' aversion to inequality: see Amiel and Creedy (1999). On the relation between the leaky bucket and the Pigou-Dalton Principle of Transfers, see Lasso de la Vega and Seidl (2007).

is the distribution function of income and Cov denotes a covariance, is:

$$G = \left(\frac{2}{\bar{x}}\right) Cov\left\{x, F\left(x\right)\right\} \tag{8}$$

In the discrete case, where incomes are in ascending order, clearly $F\left(x_i\right) = i/n$. This gave rise to an extended version of the Gini measure, $G\left(v\right)$, based on a single parameter, $v \geq 2$:

$$G\left(v\right) = -\frac{v}{\bar{x}} Cov\left\{x, \left(1 - F\left(x\right)\right)^{v-1}\right\} \tag{9}$$

A welfare rationale for this form is similar to Sen's rationale for the standard Gini.[6]

2.3 Gini and a Borda Type of Welfare Function

An important further insight into the Gini inequality measure came from taking precisely the same starting point as Atkinson, namely a form of social welfare function, W, and defining the resulting inequality measure in terms of the equally distributed equivalent income, using (4). Consider the form of W, given by:

$$W_B = \sum_{i=1}^{n} \left(n + 1 - i\right) x_i \tag{10}$$

In contrast to the form in (2), W_B is expressed as the weighted sum of incomes, again arranged in ascending order. The weights are the 'reverse ranks', $n + 1 - i$, and for this reason it is described as a Borda type of welfare function.[7] It is easily seen that the resulting equally distributed equivalent income, $x_{E,G}$ is given by:

$$x_{E,G} = \sum_{i=1}^{n} \left\{\frac{n + 1 - i}{\sum_{i=1}^{n} \left(n + 1 - i\right)}\right\} x_i \tag{11}$$

This gives the corresponding Gini-type inequality measure as:

$$G_B = 1 - \frac{x_{E,G}}{\bar{x}} \tag{12}$$

[6] Muliere and Scarsini (1989) showed that if the contribution of any v-tuple of individuals is equal to the income of the poorest person, the average social welfare of all v-tuples is $\bar{x}\left(1 - G\left(v\right)\right)$, namely the abbreviated function discussed below. For practical computations, it is not advisable to use the covariance form for $v > 2$, as the resulting $G\left(v\right)$ may not be monotonic: see Schechtman and Zitikis (2006, p. 390). On calculations, see also Schechtman and Yitzhaki (2005).

[7] The French mathematician Jean-Charles de Borda (1733–1799) proposed, in 1781, in the context of a voting system where candidates are ranked by voters, a points scoring system in which options are given scores equal to their reverse rank positions. Aggretation of scores over all voters then gives the winner as the one with the highest total score. The properties of the 'Borda Rule', and its application to a variety of aggregation settings (including committee decisions, social welfare judgements, and normative indicators such as poverty, inequality, real national income) have been intensively investigated by Sen (1977).

Close comparison of (12) with either (7) or (8) reveals that they differ slightly for small n. However, for all practical cases, with large n, they give identical values. The expression in (11) can be rearranged somewhat by recognising that the denominator involves the term, $\sum_{i=1}^{n} i = n(n+1)/2$. However, this is of marginal value from a computational point of view, and it is most useful to recognise that the weights on each x_i are simply 'normalised' reverse ranks (the reverse rank divided by the sum of reverse ranks), which sum to unity.

The interesting feature of this approach to the Gini measure is that it takes exactly the same general form as the Atkinson measure, while using a different form of $\Phi(x_i) = (n+1-i)\,x_i$. That is, it belongs to the same general class of measures: starting from an explicit expression that captures value judgements, an equally distributed equivalent income is obtained (the welfare function is easily inverted) and inequality is expressed in terms of the proportional difference between the arithmetic mean and the equally distributed equivalent income. Tibiletti and Subramanian (2015) show how this approach is also consistent with the extended Gini, $G(v)$, measures, where the weights are the powers of $(n+1-i)$.

The fact that Gini and Atkinson measures are part of the same class, involving an equally distributed equivalent income and an explicit form for the social evaluation function, means that they also share the same kind of abbreviated social welfare function, for which welfare, $\widetilde{W}$, is written as a function of arithmetic mean income and the corresponding inequality measure. The abbreviated functions for Atkinson and Gini measures are thus $\widetilde{W}_A = \bar{x}(1-I_A)$ and $\widetilde{W}_B = \bar{x}(1-G_B)$, or $\widetilde{W}_v = \bar{x}(1-G(v))$. These welfare measures actually correspond to the respective equally distributed equivalent incomes. The fact that the members of this broad class of inequality measures share the same form of abbreviated function means that they all imply the same kind of trade-off between equity (1 minus the inequality measure) and 'efficiency' (the arithmetic mean).

A different approach to evaluating welfare, not explored further here, involves starting from a given inequality measure and writing a form of abbreviated function in terms of $\bar{x}$ and inequality: this means that the starting point is an explicit trade-off between 'equity and efficiency', and an explicit link with $W(x_1, ..., x_n)$ is not considered.[8] By way of illustration, one form, involving a single parameter, α, could be the weighted geometric mean, $\widetilde{W}_{G(v),\alpha} = \bar{x}^{\alpha}(1-G(v))^{1-\alpha}$.

[8] For example, Kakwani (1980) used the form, $\widetilde{W} = \bar{x}/(1+G)$, while Dagum (1990) used $\widetilde{W} = \bar{x}(1-G)/(1+G)$. These are examined in detail in Creedy and Hurn (1999). Shorrocks (1988) used $\widetilde{W} = \bar{x}\exp(I)$, while de Graaff (1977) suggested $\widetilde{W} = \bar{x}(1-I)^{\theta}$.

3 A Further Class of Gini Measures

This section explores an additional class of Gini-type inequality measures. The starting point is the recognition that the Borda weights can be derived from a simple recurrence relation, involving a single parameter. This class is defined in subsection 3.1. The weights implied by different values of the parameter are then examined in subsection 3.2. Some examples, based on a small hypothetical income distribution, are presented in subsection 3.3.

3.1 Weights Derived From a Recurrence Relation

As mentioned in the previous section, the additional insight obtained from the Borda type of welfare function in (10) is that the Gini can be considered as simply another member, with the Atkinson type of measure, of the class of inequality indices defined in terms of an equally distributed equivalent income in relation to arithmetic mean. The value judgements involved are transparent: it is immediately clear that for any Gini measure the 'leaky bucket' thought experiment involves acceptable 'leaks' that depend only on the rank positions of transferor and transferee in the distribution. Hence, the actual incomes are not relevant, unlike the case of the Atkinson welfare function.

However, there is a further feature of the Borda weights that suggests a generalisation of the Gini to include a broader range of measures. As shown by Subramanian (2021), this arises from the recognition that the Borda weights can be obtained from a special case of a recurrence relation that is familiar to number theorists. More broadly, for the integer, k, a sequence, $f_k(i)$, for $i = 1, ..., n$, can be defined as follows. Set $f_k(1) = 1$ and $f_k(2) = Max[f_k(1), kf_k(1)]$. Then for $i = 3, ..., n$:

$$f_k(i) = kf_k(i-1) + f_k(i-2) \tag{13}$$

For the moment, consider the case where $k = 0$: further cases are discussed below. This gives the simple sequence: $1, 1, 1, 1$ and so on. The next stage involves taking the $f_k(i)$ values and producing weights to be used in the weighted sum, like the social welfare function in (10), for which the weights on incomes x_1, x_2, x_3, ..., x_n are n, $n-1, n-2, ..., 1$. Transformation of the $f_k(i)$ to these weights, denoted $S_k(i)$, is obtained using:

$$S_k(i) = \sum_{j=1}^{n+1-i} f_k(n+1-i) \tag{14}$$

Hence, for $k = 0$ and $i = 1$, this is n, obtained as the sum of n units. For $i = 2$, this is the sum of $n-1$ units, and so on. More conveniently, the expression in (14) can be rearranged as:

$$S_k(i) = \sum_{j=i}^{n} f_k(n+1-i) \tag{15}$$

Letting $S_k = \sum_{j=1}^{n} S_k(i)$, the welfare function, say W_k, can be written, by analogy with (10), as:

$$W_k = \sum_{i=1}^{n} S_k(i)\, x_i \tag{16}$$

The equally distributed equivalent income, $x_{E,k}$, becomes:

$$x_{E,k} = \sum_{i=1}^{n} \left\{ \frac{S_k(i)}{S_k} \right\} x_i \tag{17}$$

Exactly the same procedure can be used for different (integer) values of k. Hence a series of Gini measures, G_k, is given by:

$$G_k = 1 - \sum_{i=1}^{n} \left\{ \frac{S_k(i)}{S_k} \right\} \left(\frac{x_i}{\bar{x}} \right) \tag{18}$$

From a computational point of view, the expression in (18) is sufficient, and the normalised weights, $\frac{S_k(i)}{S_k}$, are easily obtained for any value of k and n. However, it is first worth briefly considering, in the following subsection, some further properties of the $f_k(i)$ sequence, for values of $k > 0$.

3.2 The Sequence $f_k(i)$ for $k > 0$

The Borda reverse rank weights $(n+1-i)$ are clearly a linear function of the rank position, i, with incomes arranged in ascending order. However, for $k > 0$ the series of weights, $S_k(i)$, is nonlinear, and the degree of nonlinearity reflects a higher aversion to inequality in the sense that higher incomes contribute relatively less to the social welfare function. For example, the sequence, $f_1(i)$, for $i = 1, ..., n$, is, from (13):

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \text{...and so on} \tag{19}$$

Furthermore, $f_2(i)$ is:

$$1, 2, 5, 12, 29.70.169, 408, 985, 2378, \text{...and so on} \tag{20}$$

These two sequences appear to diverge rapidly, but what matters for the present context are the values of the normalised weights, $S_k(i)/S_k$. Figure 1 shows, for $i = 1, 2, 3, ..., 10$, the weights produced by the Borda sequence ($k = 0$), and $k = 1, 2, 3$ and 9. The nonlinearity of the other series clearly contrasts with the weights for the Borda welfare

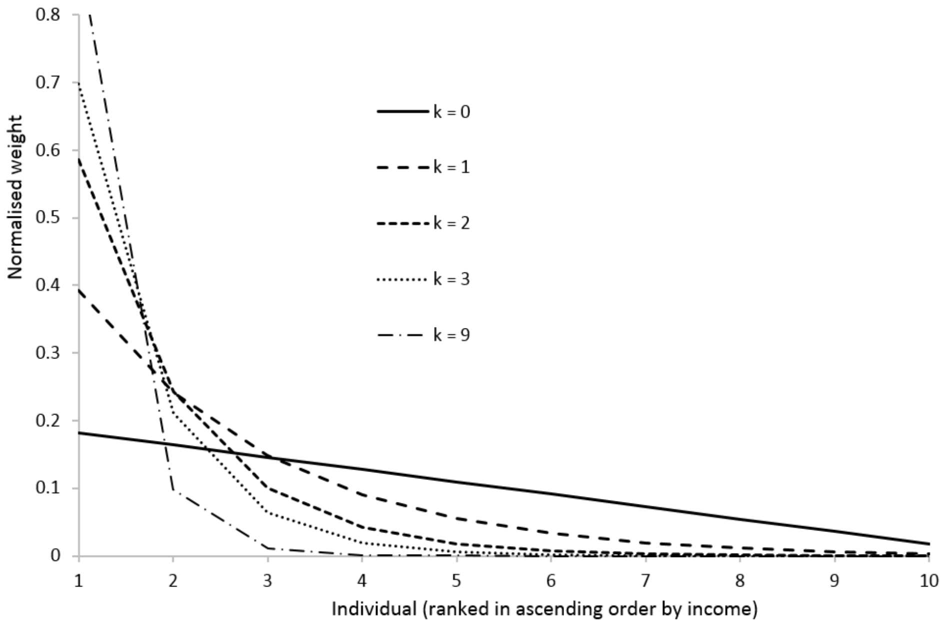

Figure 1: Welfare Weights for Alternative Series

function. For increasing k the profile approaches an L-shaped form that essentially corresponds to the Rawlsian maxi-min position.

There are additional remarkable properties of the recurrence relations. The $f_1(i)$ sequence in (19) is the famous Fibonacci series, while $f_2(i)$ in (20) is the Pell series.[9] It has long been established that each series has the property that $\frac{f_k(i)}{f_k(i-1)}$ rapidly approaches a constant. For $k = 0$, this ratio is obviously equal to 1, but for the Fibonacci series the ratio of adjacent values is the 'golden ratio', $\varphi = 1.618034$, which is the solution to $(a + b)/a = a/b$, where $a > b$. For the Pell series, the ratio rapidly approaches the 'silver ratio', $\delta = 1+\sqrt{2} = 2.414214$, and for $k = 3$, the ratio approaches the 'bronze ratio' of 3.302776.[10] The Gini measures for $k = 1, 2, 3$, and so on, may therefore be called 'metallic Ginis'.

It is possible to use the many interesting properties of these series to express the corresponding Gini measures in terms of their corresponding 'metallic ratios'. For

[9] Although it was known much earlier, the series $f_1(i)$ is named after the Italian mathematician Leonardo Bonacci (1170–approx 1240), better known simply as Fibonacci. The series $f_2(i)$ is named after the English mathematician, John Pell (1611–1685), although again it was known before Pell. It is also sometimes known as the Pell-Lucas series, after the French mathematician Francois Lucas (1842–1891).

[10] Furthermore, these are special cases of the Generalised secondary Fibonacci sequence given by: $a, b, pb + qa, p(pb + qa) + qb$, and so on, for which $f(i)/f(i-1)$ approaches $0.5\left(p + \sqrt{p^2 + 4q}\right)$.

example, Subramanian (2021) showed that the Fibonacci case becomes:

$$G_1 = \frac{\sum_{i=1}^{n} x_i \varphi^{n+3-i} - n\bar{x}\sqrt{5}}{\left\{\varphi^{n+4} - (n+3)\sqrt{5}\right\}\bar{x}} \tag{21}$$

and for the Pell series:

$$G_2 = \frac{(3\delta+1)\sqrt{2}\sum_{i=1}^{n} \delta^{n-1} x_i - 4n\bar{x}}{\{(3\delta+1)(\delta^n - 1) - 4n\}\,\bar{x}} \tag{22}$$

However, as mentioned above, from a practical computational point of view, the simple weighted sum in (18) is sufficient, especially as the same code, with only a change in the value of k, can be used to obtain all members of the class.

The connection of the Borda Gini, as simply the first in a larger class of functions involving a one-parameter recurrence relation, with these rather magical metallic ratios and their intriguing properties, is of undoubted theoretical interest. Nevertheless, it is important to establish that they also accord with various properties usually expected of inequality measures. Subramanian (2021) demonstrates, for the 'Fibonacci Index' of (21), that it satisfies the properties of symmetry, scale-invariance and replication-invariance, and transfer-sensitivity.

For these extra members to provide practical Gini alternatives to the familiar Gini measure, it is necessary to appreciate the nature of the value judgements involved, so that users can assess whether they hold similar views, and can contrast them with other value judgements. This is discussed further in Section 4. First, the following subsection provides illustrative examples using a hypothetical income distribution.

3.3 Some Examples for a Hypothetical Distribution

The different measures can usefully be illustrated using a simple hypothetical example. Consider the following incomes, arranged in ascending order, for each of 10 individuals.

$$[20, 30, 50, 55, 60, 75, 90, 120, 140, 150] \tag{23}$$

The Lorenz curve for this distribution, displaying the typical shape, is shown in Figure 2.

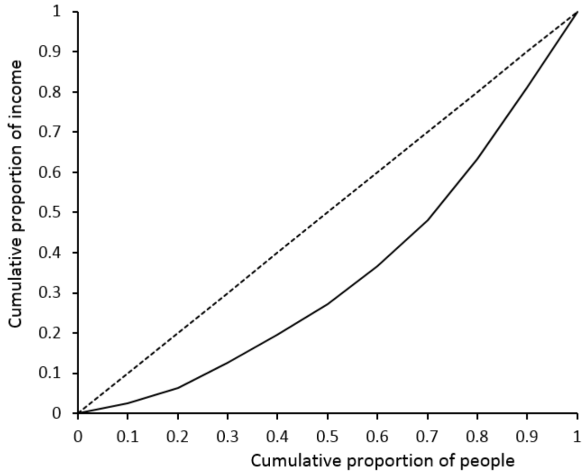

Figure 2: Lorenz Curve for Hypothetical Distribution

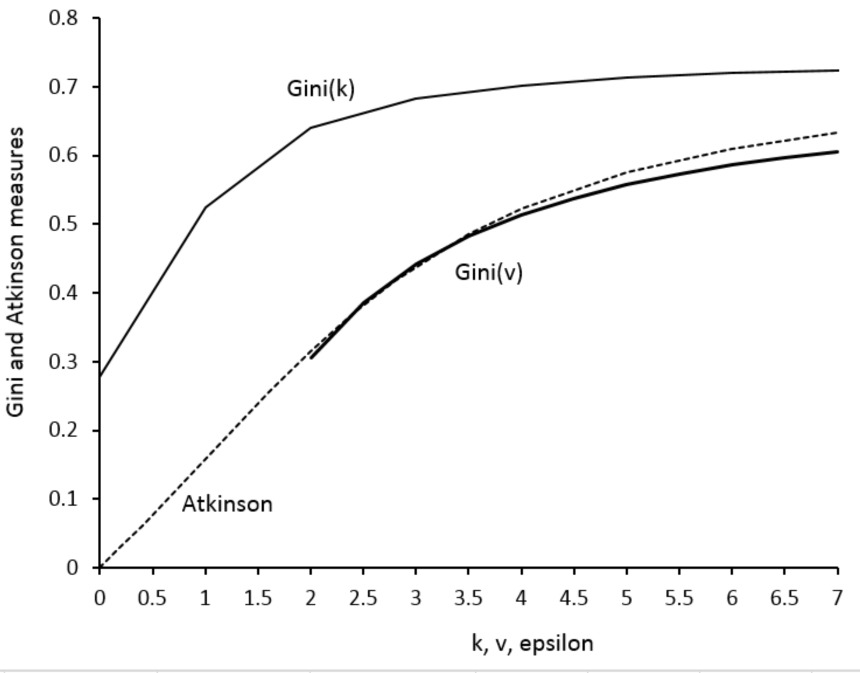

Figure 3: Alternative Gini and Atkinson Measures for Hypothetical Distribution

Figure 3 shows profiles of the members of the $G(v)$ class, along with the metallic Ginis, G_k, for variations in v and k respectively. For contrast, the diagram also shows variations in the Atkinson measure for alternative values of relative inequality aversion, ε. In each case, measured inequality is an increasing function of the single parameter associated with the class. In viewing these profile examples, it is important to recognise that they depend on the nature of the income distribution: there is, for example, no general rate of increase of $A(\varepsilon)$ with ε, or of $G(v)$ with v, or of G_k with k. If any of these measures is used to examine, say, the redistributive effect of a given tax or transfer payment (reflected in the comparison between gross and post-tax-and-transfer income), it cannot be assumed that the policy is inequality-reducing for all values of the parameter, even within a single class of measures. Thus, based on $A(\varepsilon)$, a tax may be judged to be inequality reducing for one value of ε and inequality increasing for another value of ε.

4 Value Judgements Involved in $S_k(i)$

As demonstrated by the literature on inequality measurement over the last 50 years, there is obviously nothing to prevent anyone suggesting a new class of Gini measures, by specifying some convenient function involving one or more parameters that generates a series of weights to be used in the additive social welfare function. Then, having established a number of generally desirable features (such as symmetry, continuity, mean-independence, population-neutrality, transfer and transfer-sensitivity), it is necessary to explore the way in which those implicit value judgements influence views about the tolerance for a leaky bucket, for transfers over different ranges (ranks) of the income distribution.

The class defined by the simple recurrence relation has a certain intrinsic interest and appeal, given that the existing Borda case is a special case, and the other weights are generated by famous 'metallic sequences' with their equally famous 'metallic ratios'. But their value as additional Gini measures to be considered in practice rests on whether they reflect value judgements which are likely to be held by independent judges. First, all members of the class for which $\widetilde{W} = x_E = \bar{x}(1 - I)$ share the property that the trade off between $\bar{x}$ and $1 - I$ is:

$$\left.\frac{d(1-I)}{d\bar{x}}\right|_{\widetilde{W}} = -\left(\frac{1-I}{\bar{x}}\right) \tag{24}$$

Alternatively, the elasticity of 'equity' with respect to 'efficiency' is (in absolute terms) unity, along social indifference curves.

To consider the leaky bucket, suppose there is a small transfer between two individuals, i and j. For unchanged social welfare, W_k, then:

$$\left.\frac{dx_i}{dx_j}\right|_{W_k} = -\frac{S_k(j)}{S_k(i)} \tag{25}$$

Hence if one unit is taken from person j, where $j > i$ and hence j is the richer person, the amount Δx_i must be given to i in order to keep social welfare unchanged, where:

$$\Delta x_i = \frac{S_k(j)}{S_k(i)} \tag{26}$$

The maximum 'tolerance' for leaks in making the transfer of one unit, say L_k, is thus:

$$L_k = 1 - \frac{S_k(j)}{S_k(i)} \tag{27}$$

In common with all Gini-type measures, this tolerance depends only on the rank positions of the two individuals. Using standard properties of the recurrence series, particularly involving sums of terms, it can be shown that for the Fibonacci series:

$$L_1 = 1 - \frac{\varphi^{n+3-j} - \sqrt{5}}{\varphi^{n+3-i} - \sqrt{5}} \tag{28}$$

where, as above, φ is the golden ratio. For the Pell series, involving the silver ratio, δ:

$$L_2 = 1 - \frac{\delta^{n+3-j}(3\delta+1) - 2\sqrt{2}}{\delta^{n+3-i}(3\delta+1) - 2\sqrt{2}} \tag{29}$$

To illustrate the orders of magnitude involved, consider a population consisting of just 10 individuals, as in the examples given above. Table 1 shows the leak, from one unit of income taken from the higher-ranked person, that is tolerated in making the transfer to a lower-ranked person, such that their ranks in the distribution are not affected. For example, in the standard Gini, for which $k = 0$, the maximum leak allowed by the hypothetical judge, when taking one dollar from person, $j = 2$, and transferring to person, $i = 1$, is 10 cents. When transferring to the person immediately below the transferor, the leak tolerated increases to 25 cents when $j = 8$ and $i = 7$. The maximum leak tolerated clearly increases substantially for higher values of k. For higher k, transfers to the person adjacent to the transferor are associated with similar tolerances, independent of the positions of the two persons involved. For $k = 2$, this applies also to transfers that involve large 'distances' between the ranks of those involved.

In the case of the extended Gini class of measures, $G(v)$, Creedy and Hurn (1999) show that:

$$\left.\frac{dx_i}{dx_j}\right|_{W_k} = -\frac{1 + v\left[\left(1-\frac{j}{n}\right)^{v-1} - \frac{1}{n}\sum_{s=1}^{n}\left(1-\frac{s}{n}\right)^{v-1}\right]}{1 + v\left[\left(1-\frac{i}{n}\right)^{v-1} - \frac{1}{n}\sum_{s=1}^{n}\left(1-\frac{s}{n}\right)^{v-1}\right]} \tag{30}$$

Table 1: Leak Tolerated, in cents, When Transfering One Dollar to Lower-Ranked Person: Weights From Recurrence Relation

Recipient	Transferor's rank			
	2	4	6	8
$k = 0$				
1	10	30	50	70
3		13	38	63
5			17	50
7				25
$k = 1$				
1	38	77	92	97
3		39	78	93
5			40	80
7				43
$k = 2$				
1	59	93	99	100
3		59	93	99
5			59	93
7				60

Table 2 reports, again for $n = 10$, the maximum leak tolerated (in cents) when taking a dollar from j and transferring to i ($j > i$). Of course, the values for $v = 2$ correspond to those for $k = 0$ in the previous table, and are not repeated (although there are slight differences in this small-population case).

Table 3 provides some perspective by giving the leaks tolerated for an Atkinson type of judge, with different degrees of inequality aversion, and depending on the relative incomes of transferor and transferee. In the Atkinson case, all that matters is the relative income, and in the Gini case all that matters is the rank order. The illustrations show that the different measures do indeed reflect quite different value judgements when expressed in terms of tolerance for a leaky bucket. The exception is of course for $G(2)$ and G_0, which are identical. In all cases the views, particularly regarding high relative incomes or large differences in the rank order of individuals, rapidly approach the Rawlsian maxi-min position, whereby the judge is content simply to confiscate income from the richer, or higher ranked, person.

Table 2: Leak Tolerated, in cents, When Transfering One Dollar to Lower-Ranked Person: Extended Gini

Recipient	Transferor's rank			
	2	4	6	8
$v = 4$				
1	28	66	86	93
3		33	71	86
5			35	68
7				26
$v = 6$				
1	41	81	91	93
3		42	74	78
5			27	40
7				4

Table 3: Maximum Leak Tolerated, in Cents, When Transferring One Dollar to Poorer Person: Atkinson

Ratio:	Aversion, ε			
x_j/x_i	0.2	0.8	1.5	2.0
2	13	43	65	75
4	24	67	88	94
6	30	76	93	97
8	34	81	96	98
10	37	84	97	99

5 Conclusions

This paper has explored the welfare properties of a new class of Gini income inequality measures. Instead of starting from rationales leading to relative mean difference formulations, the approach begins with a social welfare function of the Borda type, in which the income distrbution is evaluated using a weighted sum of individual incomes, and weights are equal to inverse ranks, with incomes arranged in non-descending order. This is combined with the class of inequality measures, of which the Atkinson measure is a prime example, based on the proportional difference between an equally distributed income and arithmetic mean income. The Borda case has weights that decline linearly, from the total number of individuals (the weight assigned to the lowest income) to unity (applied to the highest income). The new measures examined here are stimulated by the recognition that the Borda weights arise as a special case of a simple single-parameter recurrence relation, having a parameter of zero. The most famous case of this type of series, with a parameter of unity, is the Fibonacci series, which is known to arise in many contexts and disciplines, including biology, physics and architecture, as well as number theory. A fundamental property of each series is that the ratio of a member of the series to its preceding value quickly converges on a fixed ratio: in the Fibonacci series this is the famouse golden ratio. Other ratios in this class have been named after metals, such as silver and bronze, and for this reason the associated Gini measures may be referred to as 'metallic Ginis'.

In the case of the Atkinson measure, $A(\varepsilon)$, inequality aversion is measured in terms of the concavity of the function measuring the contribution of each income to social welfare. The iso-elastic function generally chosen has constant relative inequality aversion, ε, although this is not a fundamental feature of Atkinson's approach, which can just as easily be specified in terms of absolute, or intermediate, aversion. In the case of the extended Gini class of measures, $G(v)$, the parameter, v, is also taken to indicate the degree of aversion to inequality. In the present case of the metallic Ginis, G_k, the increasing aversion to inequality, as k is increased, is immediately indicated by the greater degree of nonlinearity in the normalised weights of the social welfare function: the weights attached to higher-ranking incomes decrease more rapidly for higher k. In all cases, increasing the relevant single parameter eventually leads to the Rawlsian maxi-min situation in which only the income of the poorest person matters in the independent judge's social evaluation.

Nevertheless, the interpretations given to the different values of ε, v or k, are far from transparent. Given that the aim of a disinterested economist is to report the implications of adopting alternative value judgements, there is a need to make as clear as possible the ways in which different values of the respective parameters reflect different attitudes towards inequality. To this end, illustrations of the 'leaky bucket' thought experiment provide useful guidance. For a specified transfer from a richer to a poorer person, the maximum leakage that would be tolerated by the judge can be evaluated. The present paper has therefore reported a range of illustrations to demonstrate the nature of aversion inherent in the metallic Ginis, and comparisons were made with Atkinson and other Gini measures. It was also shown that the maximum leakage tolerated can be related directly to the relevant 'metallic ratio'. In all cases, the maximum leakage increases as the single parameter is increased. Presented with such implied maximum leakages, for a range of types of transfer, a reader is in a stronger position to judge which inequality measure more closely accords with the reader's own value judgements. Importantly, readers also have the opportunity to conclude that none of the measures captures their views sufficiently closely.

Hence, there is no suggestion that the 'metallic Ginis' examined here are in any sense improvements on other measures, or have any special reason to be accepted as reflecting a wider range of value judgements. This is despite the finding that the standard Gini measure, that has been familiar for over a century, can be regarded 'simply' as a special case of a metallic series that contains among its members the famous golden ratio: the ratio for the standard Gini series (for $k = 0$) has a ratio of unity. What the metallic Ginis provide is an increase in the range of value judgements that can be reflected in inequality measurement. This is important because in many practical cases the ranking of different distributions does indeed depend on the nature and extent of inequality aversion.

A disappointing feature of the recent increase in the attention paid to income inequality is that the vast majority of empirical studies, particularly those from public bodies, report only one measure – typically the standard Gini – without reference to any sensitivty analysis, and with no attempt made to indicate that the use of such a measure carries with it a particular set of distributional value judgements. Given that the welfare basis of the Atkinson class of measures, and the views implicit in other measures such as the standard Gini, have been clearly established in the economics literature for about fifty years, the presentation of an additional class of measures cannot be accompanied with any optimism that they will actually be used by applied public-sector economists, and especially by those intent on making persuasive cases for policy action or inaction. But at least the metallic Ginis – like other Gini and non-Gini generalisations – provide a contribution to the literature concerned with making a wider range of sensitivity analyses possible.

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