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Analytic Valuation of GMDB Options with Utility Based Asset Allocation

Eric Ulm

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Analytic Valuation of GMDB Options with Utility Based Asset Allocation
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Abstract

A number of analytic solutions have been found for Variable Annuity Guaranteed Minimum Death Benefit (GMDB) option values under a variety of mortality laws. To date, the solutions are for Risk-Neutral valuation only. Where policyholder decisions are allowed, it is assumed that they act to maximize the risk-neutral value of the GMDB. We examine situations where the asset allocation decisions are made to maximize expected utility rather than option value. We find analytic solutions for both return of premium and ratchet options at small values of bequest motive for a number of mortality laws.

1. Introduction

There has been significant interest recently in the valuation of Guaranteed Minimum Benefits (GMxB) embedded in Variable Annuity (VA) contracts. While analytic valuations are often difficult and to date not many exist, there has been some progress made in this area for Guanteed Minimum Death Benefits (GMDB) in particular.

Milevsky and Posner (2001) find an exact solution for an at-the-money return-of-premium GMDB option under a constant force of mortality. These results were extended by Ulm (2008) to obtain exact solutions for return-of-premium and roll-up GMDB options under constant force and uniform mortality laws, as well as extending the results to all fund-to-strike ratios. Ulm (2014) found solutions for ratchet GMDB options under constant force and uniform mortality laws at all fund-to-strike ratios as well as finding solutions under the more realistic Makeham's Law of Mortality for at-the-money options.

Gerber et al (2012) use different techniques and are able to find solutions for a large array of GMDB option types beyond the simple puts analyzed previously. Gerber et al (2013) extend these results to situations where the fund follows a jump diffusion Lévy Process with exponentially distributed jumps. A formulation for the solution in terms of an integral can be found in Hardy (2003). An overview of solutions and solution methods can be found in Ulm (2019).

All of these solutions are in a risk-neutral context, assuming a complete market in which the options reside. A separate strand of literature examines the valuation of these options in a utility-based framework where one assumes the options are not tradeable but instead the option-holder acts to maximum his expected lifetime utility of consumption. To the author's knowledge no analytic solutions have been found in this context.

However, a number of numerical studies have been conducted including Gao and Ulm (2012), Moenig (2012), Moenig and Bauer (2014), Gao and Ulm (2015), Steinorth and Mitchell (2015), Moenig and Bauer (2015) and Moenig and Zhu (2018).

This paper fills the gap in analytic solutions to allocation and GMDB pricing in when the policyholder market is incomplete and he acts to maximize the lifetime utility of his VA/GMDB combination.

2. The Differential Equation to be Solved

We begin by assuming a VA contract with two fund choices, a fixed account earning a rate g and a variable account earning a market rate less fees. The variable account is assumed follow a geometric Brownian motion with an expected real-world growth rate $r_M - q$ and a volatity σ. The individual is assumed to have a CRRA utility function $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$. In the absence of a GMDB Option, the policyholder would choose to allocate a proportion of the account to the variable fund according to the Merton (1969) rule $\omega_M = \frac{r_M - q - g}{\sigma^2 \gamma}$.

We will normalize the value of the fund to make the GMDB strike $X = 1$. Since the policyholder has CRRA Utility, this is not an actual restriction on the solution. The policyholders value function $V(S,t)$ solves the following Hamilton-Jacobi-Bellman equation:

$$\frac{\partial V}{\partial t} + gS\frac{\partial V}{\partial S} + \underset{\omega(S,t)}{Max}\left[\omega(S,t)\left(r_M - q - g\right)S\frac{\partial V}{\partial S} + \frac{1}{2}\omega(S,t)^2\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]$$

$$= \left[\mu(t) + \lambda(1-\gamma) + \delta \right] V - \mu(t) \frac{b}{1-\gamma} Max[S,1]^{1-\gamma} - \lambda^{1-\gamma} \frac{S^{1-\gamma}}{1-\gamma} \qquad (1)$$

where $\mu(t)$ is the force of mortality, δ is the subjective utility discount rate, λ is the rate of withdrawal and consumption from the VA and b is the strength of the bequest motive. The derivation of this equation is found in Appendix A.

Using the verification theorem (Pham (2009) page 47), we find:

$$\omega(S,t) = \frac{-(r_M - q - g)\frac{\partial V}{\partial S}}{\sigma^2 S \frac{\partial^2 V}{\partial S^2}} \qquad (2)$$

Giving:

$$\frac{\partial V}{\partial t} + gS\frac{\partial V}{\partial S} - \frac{\left(r_M - q - g\right)^2 \left(\frac{\partial V}{\partial S}\right)^2}{2\sigma^2 \left(\frac{\partial^2 V}{\partial S^2}\right)}$$

$$= \left[\mu(t) + \lambda(1-\gamma) + \delta \right] V - \mu(t) \frac{b}{1-\gamma} Max[S,1]^{1-\gamma} - \lambda^{1-\gamma} \frac{S^{1-\gamma}}{1-\gamma} \qquad (3)$$

3. The Constant Force of Mortality Solution.

3.1 Return of Premium GMDB

3.1.1 Solution to the HJB Equation

In this case, $V(S,t)$ is independent of time. The differential equation is now:

$$gS\frac{\partial V}{\partial S}-\frac{(r_M-q-g)^2\left(\frac{\partial V}{\partial S}\right)^2}{2\sigma^2\left(\frac{\partial^2 V}{\partial S^2}\right)}$$

$$=\left[\mu(t)+\lambda(1-\gamma)+\delta\right]V-\mu(t)\frac{b}{1-\gamma}Max[S,1]^{1-\gamma}-\lambda^{1-\gamma}\frac{S^{1-\gamma}}{1-\gamma} \qquad (4)$$

An exact solution is not possible. However, we can solve the equation when $b=0$ and then find the solution for small values of b.

One can easily verify that:

$A(S)=C_A S^{1-\gamma}$;

$$C_A=\frac{\lambda^{1-\gamma}}{(1-\gamma)\left[\mu+\delta+(\lambda-g)(1-\gamma)-\frac{(r_M-q-g)^2(1-\gamma)}{2\sigma^2\gamma}\right]} \qquad (5)$$

solves the equation when $b=0$. It also satisfies the boundary condition that $V(S)$ and $\frac{\partial V}{\partial S}$ must be continuous at $S=1$.

Now suppose $V(S)=A(S)+bB(S)$. One finds that $A(S)$ must satisfy the $b=0$ equation and be given by Equation (5). Grouping the terms that are first order in b we find $B(S)$ must obey the linear differential equation:

$$gS\frac{\partial B}{\partial S}-\frac{(r_M-q-g)^2}{2\sigma^2}\left[2\frac{\partial B}{\partial S}\frac{\left(\frac{\partial A}{\partial S}\right)}{\left(\frac{\partial^2 A}{\partial S^2}\right)}-\frac{\partial^2 B}{\partial S^2}\frac{\left(\frac{\partial A}{\partial S}\right)^2}{\left(\frac{\partial^2 A}{\partial S^2}\right)^2}\right]$$

$$=\left[\mu+\lambda(1-\gamma)+\delta\right]B-\frac{\mu}{1-\gamma}Max\left[S,1\right]^{1-\gamma} \qquad (6)$$

Taking the derivatives of $A(S)$ and substituting, we find:

$$\left[g+\frac{\left(r_M-q-g\right)^2}{\sigma^2\gamma}\right]S\frac{\partial B}{\partial S}+\frac{\left(r_M-q-g\right)^2}{2\sigma^2\gamma^2}S^2\frac{\partial^2 B}{\partial S^2}$$

$$=\left[\mu+\lambda\left(1-\gamma\right)+\delta\right]B-\frac{\mu}{1-\gamma}Max\left[S,1\right]^{1-\gamma} \tag{7}$$

We find that the general solution to the homogeneous equation is $B_1S^{m_1}+B_2S^{m_2}$ where

m_1 and m_2 are the positive and negative solutions to:

$$\frac{\left(r_M-q-g\right)^2}{2\sigma^2\gamma^2}m^2+\left[g+\frac{\left(r_M-q-g\right)^2}{\sigma^2\gamma}-\frac{\left(r_M-q-g\right)^2}{2\sigma^2\gamma^2}\right]m-\left[\mu+\lambda\left(1-\gamma\right)+\delta\right]=0 \tag{8}$$

Including the specific solution to the inhomogeneous equation and choosing the exponents so that the solutions do not explode at zero and infinity we find:

$$B(S)=C_BS^{1-\gamma}+B_2S^{m_2}; \qquad S>=1$$

$$B(S)=B_0+B_1S^{m_1}; \qquad S<=1$$

$$C_B=\frac{\mu}{\left(1-\gamma\right)\left[\mu+\delta+\left(\lambda-g\right)\left(1-\gamma\right)-\frac{\left(r_M-q-g\right)^2\left(1-\gamma\right)}{2\sigma^2\gamma}\right]}$$

$$B_0=\frac{\mu}{\left[\mu+\lambda\left(1-\gamma\right)+\delta\right]\left(1-\gamma\right)} \tag{9}$$

Since $B(S)$ and $\frac{\partial B}{\partial S}$ must be continuous at $S=1$ we can show:

$$B_1=\frac{m_2B_0+(1-\gamma-m_2)C_B}{m_1-m_2} \text{ and } B_2=\frac{m_1B_0+(1-\gamma-m_1)C_B}{m_1-m_2} \tag{10}$$

Being a utility, the size of the value function is arbitrary to within an additive and multiplicative constant. However, the optimal control $\omega(S)$ is well defined and equal to:

$$\omega(S)=\omega_M\left[1+b\frac{B_1 m_1\left(\gamma+m_1-1\right)}{C_A(1-\gamma)\gamma}S^{m_1+\gamma-1}\right] \qquad S<=1$$

$$\omega(S)=\omega_M\left[1+b\frac{B_2 m_2\left(\gamma+m_2-1\right)}{C_A(1-\gamma)\gamma}S^{m_2+\gamma-1}\right] \qquad S>=1 \qquad (11)$$

to first order in b. $\omega(S)=\omega_M$ for very large and very small values of S in agreement with Gao and Ulm (2012). $\omega(S)$ is continuous at $S=1$ but is not differentiable, although it has a left hand and right hand derivative.

The solution for $V(S)$ for higher orders in b follows the same general procedure and can be continued indefinitely. Figure 1 shows the optimal value of $\omega(S)$ as a function of S for an illustrative choice of parameters. The solid line is the solution for first-order in b and the dashed line includes the second order term. Even with this relatively high bequest motive, the second order effect is not large.

The interpretation of the value of b requires some additional explanation. Equation (1) was derived assuming the beneficiary consumes the full account immediately. Assume, instead, she invests and withdraws using the same parameters as the original policyholder. The policyholder counts his beneficiary's utility at some percentage b^* relative to his own (or that of the combination of policyholder and beneficiary living off one account). In this case, the beneficiary utility is $b^*C_AS^{1-\gamma}$ and $b\equiv b^*C_A\left(1-\gamma\right)$. It is the value of b^* that is reported in the figures.

The peak in $\omega(S)$ occurs because of an "argument" between the policyholder and beneficiary first identified in Gao and Ulm (2012). The policyholder prefers the Merton allocation at all fund levels. The beneficiary prefers the Merton allocation at high fund levels because the option has little relevance to her when the put option is out of the money. Therefore, there is no argument at high fund values.

However, she prefers a strong allocation to variable accounts when the fund level is low and the option is in the money, since she is guaranteed the same payout regardless of how far the fund drops but receives a larger payout if the fund rises above the strike. Her preference, however, is not strong when the fund is well out of the money since it is quite unlikely the fund allocation will alter her results and the policyholder wins the argument. The conflict in preferences is most pronounced near the at-the-money point and this produces the peak in allocation seen in Figure 1.

3.1.2 Risk Neutral Valuation

Assuming the allocation given by Equation (11), the risk-neutral value of the option must obey the following partial differential equation, again setting the strike $X = 1$:

$$\frac{\partial V}{\partial t} + gS\frac{\partial V}{\partial S} + \omega(S,t)\left(r_f - q - g\right)S\frac{\partial V}{\partial S} + \frac{1}{2}\omega(S,t)^2\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

$$= \left(\mu(t) + r_f + \lambda\right)V - \mu(t)Max[S-1,0] \qquad (12)$$

We assume the force of mortality is constant to obtain the following differential equation:

$$\left[g + \omega(S)\left(r_f - q - g\right)\right]S\frac{\partial V}{\partial S} + \frac{1}{2}\omega(S)^2\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

$$= \left(\mu + r_f + \lambda\right)V - \mu Max[S-1,0] \qquad (13)$$

We will again assume a solution of the form $V(S) = A(S) + bB(S)$ and expand the equation in powers of b which enter through $\omega(S) = \omega_M \left[1 + bKS^{m_1+\gamma-1}\right]$ for $S <= 1$ and $\omega(S) = \omega_M \left[1 + bKS^{m_2+\gamma-1}\right]$ for $S > 1$.

We find that $A(S)$ obeys the equation:

$$\left(r_f - q^*\right) S \frac{\partial A}{\partial S} + \frac{1}{2}(\sigma^*)^2 S^2 \frac{\partial^2 A}{\partial S^2} = \left(\mu + r_f + \lambda\right) A - \mu Max[S-1,0]$$

with

$$q^* \equiv \omega_M q + \left(1 - \omega_M\right)\left(r_f - g\right)$$

$$\sigma^* \equiv \omega_M \sigma \qquad (14)$$

The solution can be found in Ulm (2006) or Ulm (2008) as:

$$A(S) = \frac{\mu}{\mu + r_f + \lambda} - \frac{\mu}{\mu + q^* + \lambda} S + A_1 S^{n_1} \qquad S <= 1$$

$$A(S) = A_2 S^{n_2} \qquad S >= 1 \qquad (15)$$

With definitions:

$$n_1 = \frac{-\left[r - q^* - \frac{(\sigma^*)^2}{2}\right] + \sqrt{\left[r - q^* - \frac{(\sigma^*)^2}{2}\right] + 2(\sigma^*)^2\left(r_f + \mu + \lambda\right)}}{(\sigma^*)^2};$$

$$n_2 = \frac{-\left[r - q^* - \frac{(\sigma^*)^2}{2}\right] - \sqrt{\left[r - q^* - \frac{(\sigma^*)^2}{2}\right] + 2(\sigma^*)^2\left(r_f + \mu + \lambda\right)}}{(\sigma^*)^2};$$

$$A_1 = \left\{\frac{\mu}{2\left(q^* + \mu + \lambda\right)}\left[1 + \frac{\xi_1}{\sqrt{\xi_1^2 + 2\left(q^* + \mu + \lambda\right)}}\right]\right.$$

$$-\frac{\mu}{2\left(r_f+\mu+\lambda\right)}\left[1+\frac{\xi_2}{\sqrt{\xi_2^2+2\left(r_f+\mu+\lambda\right)}}\right]\Bigg\};$$

$$A_2=\left\{\frac{\mu}{2\left(r_f+\mu+\lambda\right)}\left[1+\frac{\xi_2}{\sqrt{\xi_2^2+2\left(r_f+\mu+\lambda\right)}}\right]\right.$$

$$\left.-\frac{\mu}{2\left(q^*+\mu+\lambda\right)}\left[1-\frac{\xi_1}{\sqrt{\xi_1^2+2\left(q^*+\mu+\lambda\right)}}\right]\right\};$$

$$\xi_1=\frac{\left[r-q^*+\frac{(\sigma^*)^2}{2}\right]}{\sigma^*};$$

$$\xi_2=\frac{\left[r-q^*-\frac{(\sigma^*)^2}{2}\right]}{\sigma^*};\qquad (16)$$

We find that $B(S)$ obeys the equation:

$$\left(r_f-q^*\right)S\frac{\partial B}{\partial S}+\frac{1}{2}(\sigma^*)^2S^2\frac{\partial^2 B}{\partial S^2}-\left(\mu+r_f+\lambda\right)B$$
$$=-\omega_M KS^{m_1+\gamma-1}\left(r_f-q^*-g\right)S\frac{\partial A}{\partial S}-(\sigma^*)^2KS^{m_1+\gamma-1}S^2\frac{\partial^2 A}{\partial S^2}\qquad S<=1$$

$$\left(r_f-q^*\right)S\frac{\partial B}{\partial S}+\frac{1}{2}(\sigma^*)^2S^2\frac{\partial^2 B}{\partial S^2}-\left(\mu+r_f+\lambda\right)B$$
$$=-\omega_M KS^{m_2+\gamma-1}\left(r_f-q^*-g\right)S\frac{\partial A}{\partial S}-(\sigma^*)^2KS^{m_2+\gamma-1}S^2\frac{\partial^2 A}{\partial S^2}\qquad S>1\qquad (17)$$

Solving this equation and choosing the solution that satisfies the boundary conditions that both the function and its first derivative should be continuous at $S=1$ gives:

$$B(S)=B_{V1}S^{m_1+\gamma}+B_{V2}S^{n_3}+B_{V3}S^{n_1}\qquad S<=1$$

$$B(S)=B_{V4}S^{n_4}+B_{V5}S^{n_2}\qquad S>=1$$

Where

$$n_3 = n_1 + m_1 + \gamma - 1;$$

$$n_4 = n_2 + m_2 + \gamma - 1$$

$$B_{V1} = \frac{-\mu\omega_M K\left(r_f - q^* - g\right)}{\left(\mu + q^* + \lambda\right)\left[\left(r_f - q^*\right)\left(m_1 + \gamma\right) + \frac{1}{2}\left(\sigma^*\right)^2\left(m_1 + \gamma\right)\left(m_1 + \gamma - 1\right) - \left(\mu + r_f + \lambda\right)\right]};$$

$$B_{V2} = \frac{-\omega_M K A_1 n_1\left[r_f - q^* - g + \omega_M \sigma^2\left(n_1 - 1\right)\right]}{\left(r_f - q^*\right)n_3 + \frac{1}{2}\left(\sigma^*\right)^2 n_3\left(n_3 - 1\right) - \left(\mu + r_f + \lambda\right)};$$

$$B_{V4} = \frac{-\omega_M K A_2 n_2\left[r_f - q^* - g + \omega_M \sigma^2\left(n_2 - 1\right)\right]}{\left(r_f - q^*\right)n_4 + \frac{1}{2}\left(\sigma^*\right)^2 n_4\left(n_4 - 1\right) - \left(\mu + r_f + \lambda\right)};$$

$$B_{V5} = \frac{\left(n_4 - n_1\right)B_{V4} + \left(n_1 - m_1 - \gamma\right)B_{V1} + \left(n_1 - n_3\right)B_{V2}}{\left(n_1 - n_2\right)};$$

$$B_{V3} = B_{V4} + B_{V5} - B_{V1} - B_{V2}; \qquad (18)$$

This procedure can be continued in powers of the bequest motive b^2, b^3, etc. in order to produce any desired degree of accuracy in theory.

Figure 2 shows the effect of bequest motive on the value of the return of premium GMDB option. The solid line is the GMDB value with no bequest motive on the part of the policyholder[1]. The lower dashed line is the function $B(S)$ renormalized so that is multiplied by b^* (rather than b) and added to the solid line to produce the upper dashed line. The additional value of the aggressive allocation due to bequest motive peaks below

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[1] Leaving aside for the moment the issue of why a policyholder with no bequest motive would purchase a GMDB at all.

the at-the-money point. The allocation at that fund level is less aggressive, but the potential payoff is higher.

Table 1 shows the at-the-money value of the GMDB option for various parameter combinations. The increase in the GMDB value when utility-based allocation with bequest motive is introduced is substantial.

3.2 Ratchet GMDB

We now turn our attention to the solution to the HJB equation for a Ratchet GMDB. The primary difference in the solution method is in the boundary condition. The risk-neutral value at ratchet times should be proportional to the fund level at that time [see Ulm (2014)]. Analogously, in the utility-based case, the corresponding subjective utility should be $V(S) = CS^{1-\gamma}$ above the boundary.

The solution to the equation with bequest motive $b = 0$ will continue to be have the form $A(S) = C_A S^{1-\gamma}$ with C_A defined as in Equation (5). The function $B(S)$ below the boundary will continue to have the form $B(S) = B_0 + B_{1R} S^{m_1}$ with

$B_0 = \dfrac{\mu}{\left[\mu + \lambda(1-\gamma) + \delta\right](1-\gamma)}$ as in Equation (9) and A_1 to be determined from the continuity of function and derivative at the $S = X$ boundary. This leads to:

$$B(S) = \frac{\mu}{\left[\mu + \lambda(1-\gamma) + \delta\right](1-\gamma)} + \frac{\mu}{\left[\mu + \lambda(1-\gamma) + \delta\right](\gamma + m_1 - 1)} S^{m_1} \qquad (19)$$

Using $\omega(S,t) = \frac{-(r_M - q - g)\frac{\partial V}{\partial S}}{\sigma^2 S \frac{\partial^2 V}{\partial S^2}}$ we find the optimal control to be independent of time and equal to:

$$\omega(S) = \omega_M \left[1 + b\frac{m_1 B_0}{\gamma C_A} S^{m_1+\gamma-1}\right] \qquad S <= 1$$

$$\omega(S) = \omega_M \left[1 + b\frac{m_1 B_0}{\gamma C_A}\right] \qquad S >= 1 \qquad (20)$$

Figure 3 shows the optimal value of $\omega(S)$ as a function of S for an illustrative choice of parameters. The solid line is the solution for first-order in b and the dashed line is the return of premium allocation for comparison. The ratchet allocation is not shown for $S > 1$ since this situation will not occur in reality. Surprisingly, the ratchet benefit does not substantially increase the optimal allocation to the variable account.

The risk-neutral valuation proceeds analogously to the return-of-premium option valuation in Section 3.1.2. We again assume the risk-neutral value $V(S) = A(S) + bB(S)$ satisfies Equations (13) and (17) below the boundary and $V(S) \propto S$ above the boundary. The solution below the boundary can be found as:

$$A(S) = \frac{\mu}{\mu + r_f + \lambda} - \frac{\mu}{\mu + q^* + \lambda} S + \frac{\mu}{(\mu + r_f + \lambda)(n_1 - 1)} S^{n_1} \qquad (21)$$

as in Ulm (2014), and

$$B(S) = B_{R1} S + B_{R2} S^{n_3} + B_{R3} S^{n_1}$$

With

$$B_{R1}=\frac{-\mu\omega_M m_1 B_0\left(r_f-q-g\right)}{\gamma C_A\left(\mu+q^*+\lambda\right)\left[\left(r_f-q^*\right)\left(m_1+\gamma\right)+\frac{1}{2}\left(\sigma^*\right)^2\left(m_1+\gamma\right)\left(m_1+\gamma-1\right)-\left(\mu+r_f+\lambda\right)\right]};$$

$$B_{R2}=\frac{-m_1 B_0\omega_M\mu\left[r_f-q-g+\omega_M\sigma^2\left(n_1-1\right)\right]}{\gamma C_A\left(\mu+r_f+\lambda\right)\left(n_1-1\right)\left[\left(r_f-q^*\right)n_3+\frac{1}{2}\left(\sigma^*\right)^2 n_3\left(n_3-1\right)-\left(\mu+r_f+\lambda\right)\right]};$$

$$B_{R3}=\frac{\left(1-n_3\right)B_{R2}+\left(1-m_1-\gamma\right)B_{R1}}{n_1-1}; \tag{22}$$

Figure 4 shows the effect of bequest motive on the value of the return of premium GMDB option. The solid line is the GMDB value with no bequest motive on the part of the policyholder. The lower dashed line is the function $B(S)$ renormalized so that is multiplied by b^* (rather than b) and added to the solid line to produce the upper dashed line. Table 2 shows the at-the-money value of the GMDB option for various parameter combinations. The increase in the GMDB value when utility-based allocation with bequest motive is introduced is again substantial.

3.3. Optimal Consumption

We now revisit the solution to the HJB Equation [Equation (1)] assuming that the withdrawal rate λ is a second control variable. The verification theorem suggests:

$$\lambda\left(S,t\right)=\left[\left(1-\gamma\right)V\left(S,t\right)S^{\gamma-1}\right]^{-\frac{1}{\gamma}} \tag{23}$$

The HJB Equation becomes:

$$\frac{\partial V}{\partial t}+gS\frac{\partial V}{\partial S}-\frac{\left(r_M-q-g\right)^2\left(\frac{\partial V}{\partial S}\right)^2}{2\sigma^2\left(\frac{\partial^2 V}{\partial S^2}\right)}$$

$$=\left[\mu(t)+\delta\right]V-\mu(t)\frac{b}{1-\gamma}Max[S,1]^{1-\gamma}-\gamma\left(1-\gamma\right)^{-\frac{1}{\gamma}}V^{\frac{\gamma-1}{\gamma}}S^{\frac{1-\gamma}{\gamma}} \tag{24}$$

Under constant force of mortality, this becomes:

$$gS\frac{\partial V}{\partial S}-\frac{\left(r_M-q-g\right)^2\left(\frac{\partial V}{\partial S}\right)^2}{2\sigma^2\left(\frac{\partial^2 V}{\partial S^2}\right)}$$

$$=\left(\mu+\delta\right)V-\mu(t)\frac{b}{1-\gamma}Max[S,1]^{1-\gamma}-\gamma\left(1-\gamma\right)^{-\frac{1}{\gamma}}V^{\frac{\gamma-1}{\gamma}}S^{\frac{1-\gamma}{\gamma}} \tag{25}$$

Letting $V(S)=A(S)+bB(S)$ We again find:

$A(S)=C_A S^{1-\gamma}$;

$$C_A=\frac{1}{\left(1-\gamma\right)}\left[\frac{\gamma}{\left[\mu+\delta-g\left(1-\gamma\right)-\frac{\left(r_M-q-g\right)^2\left(1-\gamma\right)}{2\sigma^2\gamma}\right]}\right]^{\gamma} \tag{26}$$

If we define:

$$\lambda_M=\frac{\left[\mu+\delta-g\left(1-\gamma\right)-\frac{\left(r_M-q-g\right)^2\left(1-\gamma\right)}{2\sigma^2\gamma}\right]}{\gamma} \tag{27}$$

then C_A is as defined in Equation (5) with $\lambda\to\lambda_M$.

We also find:

$$\left[g+\frac{\left(r_M-q-g\right)^2}{\sigma^2\gamma}\right]S\frac{\partial B}{\partial S}+\frac{\left(r_M-q-g\right)^2}{2\sigma^2\gamma^2}S^2\frac{\partial^2 B}{\partial S^2}$$

$$=\left[\mu+\lambda_M\left(1-\gamma\right)+\delta\right]B-\frac{\mu}{1-\gamma}Max\left[S,1\right]^{1-\gamma} \qquad (28)$$

which is identical to Equation (7) with $\lambda \to \lambda_M$ and has the same solution. The optimal allocation $\omega(S)$ follows Equation (11) with $\lambda \to \lambda_M$. The optimal withdrawal is:

$$\lambda=\left\{\left(1-\gamma\right)\left[A\left(S\right)+bB\left(S\right)\right]S^{\gamma-1}\right\}^{-\frac{1}{\gamma}}=\lambda_M\left[1+b\frac{B\left(S\right)}{A\left(S\right)}\right]^{-\frac{1}{\gamma}} \qquad (29)$$

Equation (29) holds for both Return-of-Premium and Ratchet GMDB options as long as $B(S)$ is the function corresponding to the appropriate option type.

Figure 5 shows the optimal withdrawal pattern for representative parameters. When the risk-aversion parameter is less than 1, withdrawals are lower when the option is in the money. When the risk-aversion parameter is greater than 1, withdrawals are higher when the option is in the money. This same pattern was found in Gao and Ulm (2012) and has the same explanation. That is, when $\gamma > 1$, the beneficiary's utility is actually reduced less by a withdrawal than the policyholder's utility is reduced. The policyholder chooses additional consumption at low fund values since this does not harm the beneficiary as much as it helps the policyholder.

Risk-Neutral Valuation follows the same procedure as in Section 3.1.2. We can expand λ in powers of b to get, in the return of premium case:

$$\lambda(S)=\lambda_M\left\{1-b\left[\frac{B_0}{\gamma C_A}S^{\gamma-1}+\frac{B_1}{\gamma C_A}S^{m_1+\gamma-1}\right]\right\} \qquad S<=1$$

$$\lambda(S) = \lambda_M \left\{ 1 - b\left[\frac{C_B}{\gamma C_A} + \frac{B_2}{\gamma C_A} S^{m_2+\gamma-1} \right] \right\} \qquad S >= 1 \qquad (30)$$

Letting $V(S) = A(S) + bB(S)$ We again find $A(S)$ as described in Equation (15) with $\lambda \to \lambda_M$. The equation for $B(S)$ involves quite a few additional terms relative to Equation (17), but can be solved as:

$$B(S) = B_{V1}S^{m_1+\gamma} + B_{V2}S^{n_3} + B_{V3}S^{m_1+\gamma-1} + B_{V4}S^{\gamma-1}$$

$$+B_{V5}S^{\gamma} + B_{V6}S^{n_1+\gamma-1} + B_{V7}S^{n_1} \qquad S <= 1$$

$$B(S) = B_{V8}S^{n_4} + B_{V9}S^{n_2}\ln S + B_{V10}S^{n_2} \qquad S >= 1$$

Where

$$n_3 = n_1 + m_1 + \gamma - 1;$$

$$n_4 = n_2 + m_2 + \gamma - 1$$

$$B_{V1} = \frac{-\mu\left[\omega_M K\left(r_f - q^* - g\right) + \frac{\lambda_M B_1}{\gamma C_A}\right]}{\left(\mu + q^* + \lambda_M\right)\left[\left(r_f - q^*\right)\left(m_1 + \gamma\right) + \frac{1}{2}\left(\sigma^*\right)^2\left(m_1 + \gamma\right)\left(m_1 + \gamma - 1\right) - \left(\mu + r_f + \lambda_M\right)\right]};$$

$$B_{V2} = \frac{-\left\{\omega_M K A_1 n_1\left[r_f - q^* - g + \omega_M \sigma^2\left(n_1 - 1\right)\right] + \frac{\lambda_M B_1}{\gamma C_A} A_1\right\}}{\left(r_f - q^*\right)n_3 + \frac{1}{2}\left(\sigma^*\right)^2 n_3\left(n_3 - 1\right) - \left(\mu + r_f + \lambda_M\right)};$$

$$B_{V3} = \frac{\dfrac{\lambda_M B_1 \mu}{\gamma C_A\left(\mu + r_f + \lambda_M\right)}}{\left[\left(r_f - q^*\right)\left(m_1 + \gamma - 1\right) + \frac{1}{2}\left(\sigma^*\right)^2\left(m_1 + \gamma - 1\right)\left(m_1 + \gamma - 2\right) - \left(\mu + r_f + \lambda_M\right)\right]};$$

$$B_{V4} = -\frac{\dfrac{\lambda_M B_0 \mu}{\gamma C_A\left(\mu + r_f + \lambda_M\right)}}{\left[\left(r_f - q^*\right)\left(\gamma - 1\right) + \frac{1}{2}\left(\sigma^*\right)^2\left(\gamma - 1\right)\left(\gamma - 2\right) - \left(\mu + r_f + \lambda_M\right)\right]};$$

$$B_{V5} = \frac{\dfrac{\lambda_M B_0 \mu}{\gamma C_A\left(\mu + q^* + \lambda_M\right)}}{\left[\left(r_f - q^*\right)\gamma + \frac{1}{2}\left(\sigma^*\right)^2\gamma\left(\gamma - 1\right) - \left(\mu + r_f + \lambda_M\right)\right]};$$

$$B_{V6} = -\frac{\dfrac{\lambda_M B_0 A_1}{\gamma C_A}}{\left[\left(r_f - q^*\right)\left(n_1 + \gamma - 1\right) + \frac{1}{2}\left(\sigma^*\right)^2\left(n_1 + \gamma - 1\right)\left(n_1 + \gamma - 2\right) - \left(\mu + r_f + \lambda_M\right)\right]};$$

$$B_{V8} = \frac{-\left\{\omega_M K A_2 n_2\left[r_f - q^* - g + \omega_M \sigma^2\left(n_2 - 1\right)\right] + \dfrac{\lambda_M B_2}{\gamma C_A} A_2\right\}}{\left(r_f - q^*\right)n_4 + \frac{1}{2}\left(\sigma^*\right)^2 n_4\left(n_4 - 1\right) - \left(\mu + r_f + \lambda_M\right)};$$

$$B_{V9} = -\frac{\dfrac{\lambda_M C_B A_1}{\gamma C_A}}{\left[\left(\sigma^*\right)^2 n_2 - \left(r_f - q^* - \frac{1}{2}\left(\sigma^*\right)^2\right)\right]}$$

$$B_{V10} = \frac{\left(n_4 - n_1\right)B_{V8} + B_{V9} + \left(n_1 - m_1 - \gamma\right)B_{V1} + \left(n_1 - n_3\right)B_{V2} + \left(n_1 - m_1 - \gamma + 1\right)B_{V3}}{\left(n_1 - n_2\right)}$$

$$+ \frac{\left(n_1 - \gamma + 1\right)B_{V4} + \left(n_1 - \gamma\right)B_{V5} + \left(1 - \gamma\right)B_{V6}}{\left(n_1 - n_2\right)};$$

$$B_{V7} = B_{V8} + B_{V10} - B_{V1} - B_{V2} - B_{V3} - B_{V4} - B_{V5} - B_{V6}; \qquad (31)$$

Figures 6 and 7 show the risk-neutral option value for some representative parameters. When $\gamma = 0.7$, the additional component $B(S)$ increases near $S = 0$ since lapses are reduced there. When $\gamma = 1.8$, the additional component $B(S)$ is zero near

$S = 0$ since lapses are increased there. There is also a increase in $B(S)$ near the at-the-money point due to the effect of the more aggressive allocation. This is more clearly visible in Figure 7 than in Figure 6.

The valuation of the ratchet GMDB follows similarly. $A(S)$ as described in Equation (15) with $\lambda \to \lambda_M$. $B(S)$ is defined as in Equation (31) for $S <= 1$. The coefficients B_{V1} through B_{V6} are defined as in Equation (31) with

$$A_1 \to A_{R1} = \frac{\mu}{\left(\mu + r_f + \lambda_M\right)\left(n_1 - 1\right)} \text{ and } B_1 \to B_{1R} = \frac{B_0\left(1-\gamma\right)}{\left(\gamma + m_1 - 1\right)}.$$

$$B_{V7} = \frac{\left(1-m_1-\gamma\right)B_{V1} + \left(1-n_3\right)B_{V2} + \left(2-m_1-\gamma\right)B_{V3}}{\left(n_1-1\right)} + \frac{\left(2-\gamma\right)B_{V4} + \left(1-\gamma\right)B_{V5} + \left(2-n_1-\gamma\right)B_{V6}}{\left(n_1-1\right)}; \tag{32}$$

4. The DeMoivre's Law Solution

4.1 Return of Premium GMDB

We start our analysis from Equation (4) an again require $V(S) = A(S) + bB(S)$ for small b. If we let:

$$A(S) = a\left(t\right)S^{1-\gamma} \tag{33}$$

we find $a\left(t\right)$ obeys the equation:

$$a'\left(t\right) + \left[\left(1-\gamma\right)g + \frac{\left(r_M - q - g\right)^2}{2\sigma^2}\left(\frac{1-\gamma}{\gamma}\right) - \mu(t) - \lambda\left(1-\gamma\right) - \delta\right]a\left(t\right) = -\frac{\lambda^{1-\gamma}}{1-\gamma} \tag{34}$$

Expressing Equation (34) in the form of Thiele's Differential Equation we can show:

$$a(t) = \frac{\lambda^{1-\gamma}}{1-\gamma}\bar{a}_{x+t} \tag{35}$$

where the annuity symbol is evaluated at rate $r_{ann} = \delta + \left(\lambda - g - \frac{(r_M - q - g)^2}{2\sigma^2\gamma} \right)(1-\gamma)$.

Equation (33)-(35) are valid for any general mortality law, not just DeMoivre's Law. We also find, for any general mortality law, that:

$$\frac{\partial B}{\partial t} + \left[g + \frac{(r_M - q - g)^2}{\sigma^2\gamma} \right] S \frac{\partial B}{\partial S} + \frac{(r_M - q - g)^2}{2\sigma^2\gamma^2} S^2 \frac{\partial^2 B}{\partial S^2}$$

$$= \left[\mu(t) + \lambda(1-\gamma) + \delta \right] B - \frac{\mu(t)}{1-\gamma} Max\left[S,1\right]^{1-\gamma} \tag{36}$$

This is extremely similar to Equation (3) in Ulm (2008) with the definitions $r^{**} = \delta$, $q^{**} = \delta - g - (\omega_M \sigma)^2$, $\sigma^{**} = \omega_M \sigma$ and a slightly different driving term. It is solved in the same manner, giving:

Region I. $S > 1$

$$B(S,t) = \frac{1}{1-\gamma}\left[\frac{1-e^{-r_{LI}(T-t)}}{r_{LI}(T-t)} S^{1-\gamma} + \frac{C_1}{(T-t)} S^{m_1} N(-\xi_1) + \frac{C_2}{(T-t)} S^{m_2} N(-\xi_2) \right.$$

$$\left. - \frac{e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})(T-t)} N(-d_2) + \frac{e^{-r_{LI}(T-t)}}{r_{LI}(T-t)} S^{1-\gamma} N(-d_1) \right]$$

and

Region II. $S < 1$

$$B(S,t)=\frac{1}{1-\gamma}\left[\frac{1-e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})(T-t)}-\frac{C_1}{(T-t)}S^{m_1}N(\xi_1)-\frac{C_2}{(T-t)}S^{m_2}N(\xi_2)\right.$$

$$\left.+\frac{e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})(T-t)}N(d_2)-\frac{e^{-r_{LI}(T-t)}}{r_{LI}(T-t)}S^{1-\gamma}N(d_1)\right] \tag{37}$$

with definitions:

$$r_{LI}=r_{ann}+\frac{1}{2}\sigma^{**2}\left(1-\gamma\right)\gamma$$

$$d_1=\frac{\ln S+\left(r^{**}-q^{**}+\frac{\sigma^{**2}}{2}-\gamma\sigma^{**}\right)(T-t)}{\sigma^{**}\sqrt{(T-t)}};$$

$$d_2=\frac{\ln S+\left(r^{**}-q^{**}-\frac{\sigma^{**2}}{2}\right)(T-t)}{\sigma^{**}\sqrt{(T-t)}};$$

$$\xi_1=\frac{\ln S+\sigma^{**2}\sqrt{K}(T-t)}{\sigma^{**}\sqrt{(T-t)}};$$

$$\xi_2=\frac{\ln S-\sigma^{**2}\sqrt{K}(T-t)}{\sigma^{**}\sqrt{(T-t)}};$$

$$m_1=A+\sqrt{K};$$

$$m_2=A-\sqrt{K};$$

$$C_1=\frac{1}{2r_{LI}}\left[\frac{A+\gamma-1}{\sqrt{K}}-1\right]-\frac{1}{2(r^{**}+\lambda^{**})}\left[\frac{A}{\sqrt{K}}-1\right];$$

$$C_2 = \frac{1}{2(r^{**}+\lambda^{**})}\left[1+\frac{A}{\sqrt{K}}\right]-\frac{1}{2r_{LI}}\left[1+\frac{A+\gamma-1}{\sqrt{K}}\right];$$

$$A = \frac{1}{2}-\frac{(r^{**}-q^{**})}{\sigma^{**2}};$$

$$K = A^2+\frac{2(r^{**}+\lambda^{**})}{\sigma^{**2}} \tag{38}$$

Again, being a utility, the size of the value function is arbitrary to within an additive and multiplicative constant. However, the optimal control $\omega(S)$ is well defined from Equation (2). Continuing the small b approximation we find, for all mortality laws:

$$\omega(S,t) \approx \omega_M\left[1+b\frac{S^\gamma}{a(t)(1-\gamma)}\left(\frac{\partial B}{\partial S}+\frac{S}{\gamma}\frac{\partial^2 B}{\partial S^2}\right)\right] \tag{39}$$

Equation (39) can be shown to reproduce Equation (11) for the constant force of mortality case. When mortality follows DeMoivre's Law, the partial derivatives can be done in closed form by differentiating Equation (38).

Figure 8 shows the optimal allocation $\omega(S,t)$ as a function of S for a representative choice of parameters. As in Figure 1, the beneficiary is assumed to be an optimizing individual with a constant force of mortality $\mu = 0.01$. The DeMoivre's Law mortality rate is quite close to that used in Figure 1 and the allocation is therefore also quite similar.

Figure 9 shows the optimal allocation $\omega(S,t)$ as a function of S for the same choice of parameters and a maximum age of 120. The at-the-money allocation approaches infinity as the age of the policyholder approaches the maximum as the beneficiary is now almost certain to get the money and wins the argument in a landslide.

However, this effect would be mitigated in practice since $\omega\left(S,t\right)>1$ would not be possible for real world GMDB contracts.

4.2 Ratchet GMDB

The solution for the Ratchet GMDB also solves Equations (33)-(36), although the boundary condition is different. The boundary condition on $B\left(S,t\right)$ is now

$$\frac{\partial B}{\partial S}=\left(1-\gamma\right)B \tag{40}$$

The solution for $A\left(S,t\right)$ is the same as in the Return of Premium case in Section 4.1. The differential equation to be solved for $B\left(S,t\right)$ is quite similar to Equations (1) and (2) in Ulm (2014) with the definitions $r^{**}=\delta$, $q^{**}=\delta-g-\left(\omega_M\sigma\right)^2$, $\sigma^{**}=\omega_M\sigma$ with a slightly different driving term and boundary conditions. It is solved in the same manner, giving:

$$B(S,t)=\frac{1-e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})\left(1-\gamma\right)(T-t)}+\frac{\sigma^{**2}}{(r^{**}+\lambda^{**})\left(m_1+\gamma-1\right)(T-t)}S^{m_1}N(\xi_1)$$

$$+\frac{\sigma^{**2}}{(r^{**}+\lambda^{**})\left(m_2+\gamma-1\right)(T-t)}S^{m_1}N(\xi_2)$$

$$-\frac{2\left(r^{**}-q^{**}\right)+\left(1-2\gamma\right)\sigma^{**2}}{\left[2\left(r^{**}-q^{**}\right)-\gamma\sigma^{**2}\right]\left(1-\gamma\right)}\frac{e^{-r_{LI}(T-t)}}{r_{LI}(T-t)}S^{1-\gamma}N(d_1)$$

$$+\frac{e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})(1-\gamma)(T-t)}N(d_2)$$

$$+\frac{\sigma^{**2}}{2\left(r^{**}-q^{**}\right)-\gamma\sigma^{**2}}\frac{e^{-(r^{**}+\lambda^{**})(T-t)}}{(r^{**}+\lambda^{**})(T-t)}S^{2\alpha}N(d_3)\Bigg] \quad (41)$$

with definitions as in Equation (38) and

$$d_3=\frac{\ln S+\left[-\left(r^{**}-q^{**}\right)+\frac{\sigma^{**2}}{2}\right](T-t)}{\sigma^{**}\sqrt{(T-t)}} \quad (42)$$

We can again find the optimal allocation from Equation (39). Figure 10 shows the optimal allocation $\omega(S,t)$ as a function of S for a representative choice of parameters. As in Figure 8, the beneficiary is assumed to be an optimizing individual with a constant force of mortality $\mu=0.01$. The optimal allocation is not substantially different from the Return of Premium optimal allocation, as was the case for constant force of mortality. Figure 11 shows the optimal allocation $\omega(S,t)$ as a function of S for the same choice of parameters and a maximum age of 120. Unfortunately, the PDEs corresponding to the valuation equation and the optimal consumption equations are too complicated and have resisted attempts at solutions.

5. Conclusions

In this paper, we derive closed form solutions for optimal allocation and consumption patterns in Variable Annuities with GMDB riders. In agreement with the numerical study of Gao and Ulm (2012), the optimal allocation peaks near the at-the-money point and significantly affects the value of the option. In addition, new results are derived for Ratchet GMDB options and it is shown that the optimal allocation in this case is not significantly different than the allocation obtained for Return of Premium options.

6. Acknowledgements

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Appendix A

Derivation of the HJB Equation for GMDB options with CRRA Utility.

We find the relevant equation by taking the limit of the finite time equation as $\Delta t \to 0$. At any time t, fund level S and strike X, the value of the fund is given recursively by:

$$V(S,X,t) = U\left[c(S,X,t)\right]\Delta t + \mu(t)bU\left[DB\right]\Delta t$$

$$+\left(1-\mu(t)\Delta t\right)\beta^{\Delta t}\int_{-\infty}^{\infty} V\left(w, X - c\frac{X}{S}\Delta t, t+\Delta t\right)\Phi\left(w, S + rS\Delta t - c\Delta t, \sigma S\sqrt{\Delta t}\right)dw \quad \text{(A1)}$$

Where $U[c]$ is the utility function for consumption, $\mu(t)$ is the mortality rate, β is the subjective discount factor and $\Phi(w, Mean, StDev)$ is the pdf of the normal distribution. We assume the time step is small enough that the lognormal distribution can be approximated by a normal distribution.

Let $D(w) = V(w, X - c\frac{X}{S}\Delta t, t+\Delta t)$. A Taylor expansion gives:

$$D(w) = D\left(S + rS\Delta t - c\Delta t\right) + \left.\frac{\partial D}{\partial w}\right|_{S+rS\Delta t - c\Delta t}\left[w - \left(S + rS\Delta t - c\Delta t\right)\right]$$

$$+\frac{1}{2}\left.\frac{\partial^2 D}{\partial w^2}\right|_{S+rS\Delta t - c\Delta t}\left[w - \left(S + rS\Delta t - c\Delta t\right)\right]^2 \quad \text{(A2)}$$

Also, let $\delta = -\ln\beta$; $x = w - \left(S + rS\Delta t - c\Delta t\right)$ and

$\Phi\left(x, 0, \sigma S\sqrt{\Delta t}\right) = \Phi\left(w, S + rS\Delta t - c\Delta t, \sigma S\sqrt{\Delta t}\right)$. This gives:

$$V(S,X,t) = U\left[c(S,X,t)\right]\Delta t + \mu(t)bU\left[DB\right]\Delta t$$

$$+\left[1-\left(\mu(t)+\delta\right)\Delta t\right]\int_{-\infty}^{\infty}\left[D\left(S+rS\Delta t-c\Delta t\right)+\left.\frac{\partial D}{\partial w}\right|_{S+rS\Delta t-c\Delta t}x\right.$$

$$\left.+\frac{1}{2}\left.\frac{\partial^2 D}{\partial w^2}\right|_{S+rS\Delta t-c\Delta t}x^2\right]\Phi\left(x,0,\sigma S\sqrt{\Delta t}\right)dx \qquad \text{(A3)}$$

The integral resolves to give:

$$V(S,X,t)=U\left[c\left(S,X,t\right)\right]\Delta t+\mu(t)bU\left[DB\right]\Delta t$$

$$+\left[1-\left(\mu(t)+\delta\right)\Delta t\right]\left[D\left(S+rS\Delta t-c\Delta t\right)+\frac{1}{2}\left.\frac{\partial^2 D}{\partial w^2}\right|_{S+rS\Delta t-c\Delta t}\sigma^2 S^2\Delta t\right] \qquad \text{(A4)}$$

Now,

$$\left.\frac{\partial^2 D}{\partial w^2}\right|_{S+rS\Delta t-c\Delta t}=\left.\frac{\partial^2 V}{\partial S^2}\right|_{S,X,t}+o\left(\Delta t^2\right)$$

And

$$D\left(S+rS\Delta t-c\Delta t\right)=V(S+rS\Delta t-c\Delta t,X-c\frac{X}{S}\Delta t,t+\Delta t)$$
$$=V(S,X,t)+\frac{\partial V}{\partial S}\left(rS-c\right)\Delta t-\frac{\partial V}{\partial X}\left(c\frac{X}{S}\right)\Delta t+\frac{\partial V}{\partial t}\Delta t+o\left(\Delta t^2\right)$$

Giving:

$$\frac{\partial V}{\partial t}+\left(rS-c\right)\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-\left(c\frac{X}{S}\right)\frac{\partial V}{\partial X}$$

$$=\left[\mu\left(t\right)+\delta\right]V-U\left[c\left(S,X,t\right)\right]-\mu\left(t\right)bU\left(DB\right) \qquad \text{(A5)}$$

We will now normalize by dividing by the strike and using the properties of a CRRA utility function. Define $V^*\left(S,t\right)=V\left(S,1,t\right)$, $S^*=\frac{S}{X}$ and $c^*=\frac{c}{X}$. Using the scaling properties of a CRRA Utility function, $V\left(S,X,t\right)=X^{1-\gamma}V\left(\frac{S}{X},1,t\right)=X^{1-\gamma}V^{*}\left(S^{*},t\right)$.

Therefore:

$$\frac{\partial V}{\partial t}=X^{1-\gamma}\frac{\partial V^{*}}{\partial t};$$

$$\frac{\partial V}{\partial S}=X^{1-\gamma}\frac{\partial V^{*}}{\partial S^{*}}\frac{\partial S^{*}}{\partial S}=X^{1-\gamma}\frac{\partial V^{*}}{\partial S^{*}}\left(\frac{1}{X}\right);$$

$$\frac{\partial^{2}V}{\partial S^{2}}=X^{1-\gamma}\left(\frac{1}{X}\right)\frac{\partial^{2}V^{*}}{\partial S^{*2}}\frac{\partial S^{*}}{\partial S}=X^{1-\gamma}\frac{\partial V^{*}}{\partial S^{*}}\left(\frac{1}{X^{2}}\right)$$

$$\frac{\partial V}{\partial X}=\left(1-\gamma\right)X^{-\gamma}V^{*}+X^{1-\gamma}\frac{\partial V^{*}}{\partial S^{*}}\frac{\partial S^{*}}{\partial X}=\left(1-\gamma\right)X^{-\gamma}V^{*}+X^{1-\gamma}\frac{\partial V^{*}}{\partial S^{*}}\left(\frac{-S}{X^{2}}\right)$$
$$=X^{1-\gamma}\left[\left(1-\gamma\right)\frac{V^{*}}{X}-\frac{S}{X^{2}}\frac{\partial V^{*}}{\partial S^{*}}\right]$$

which gives:

$$\frac{\partial V^{*}}{\partial t}+\left(rS^{*}-c^{*}\right)\frac{\partial V^{*}}{\partial S^{*}}+\frac{1}{2}\sigma^{2}S^{*2}\frac{\partial^{2}V^{*}}{\partial S^{*2}}+c^{*}\frac{\partial V^{*}}{\partial S^{*}}$$
$$=\left[\mu\left(t\right)+\delta+\frac{c^{*}}{S^{*}}(1-\gamma)\right]V^{*}-\frac{c^{*(1-\gamma)}}{1-\gamma}-\mu\left(t\right)b\frac{Max(S^{*},1)^{1-\gamma}}{1-\gamma} \tag{A5}$$

Finally, we define the withdrawal rate $\lambda=\frac{c^{*}}{S^{*}}$ and remove the asterisks from the variables to get:

$$\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}V}{\partial S^{2}}$$
$$=\left[\mu\left(t\right)+\delta+\lambda(1-\gamma)\right]V-\lambda^{1-\gamma}\frac{S^{(1-\gamma)}}{1-\gamma}-\mu\left(t\right)b\frac{Max(S^{*},1)^{1-\gamma}}{1-\gamma} \tag{A6}$$

which is text Equation (1) with return $r=\left(1-\omega\right)g+\omega\left(r_{M}-q\right)$ and volatility $\omega\sigma$.

Figure 1
Optimal Allocation for Return of Premium GMDB Options with a Constant Force of Mortality

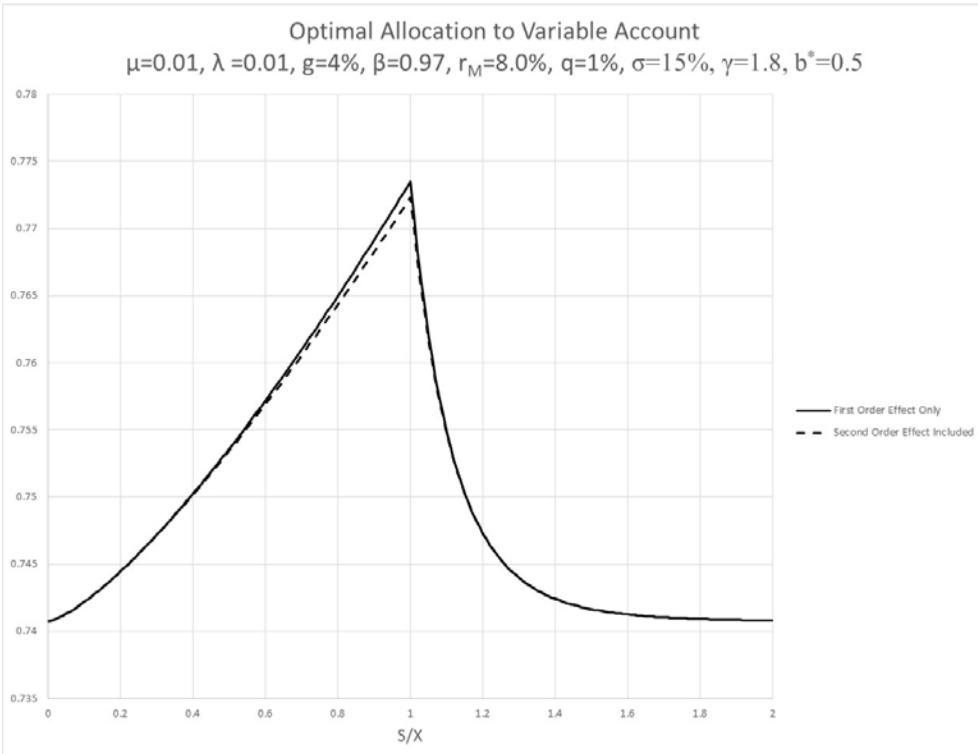

Figure 2
Risk Neutral Value of Return of Premium GMDB Option
With and Without Bequest Motive

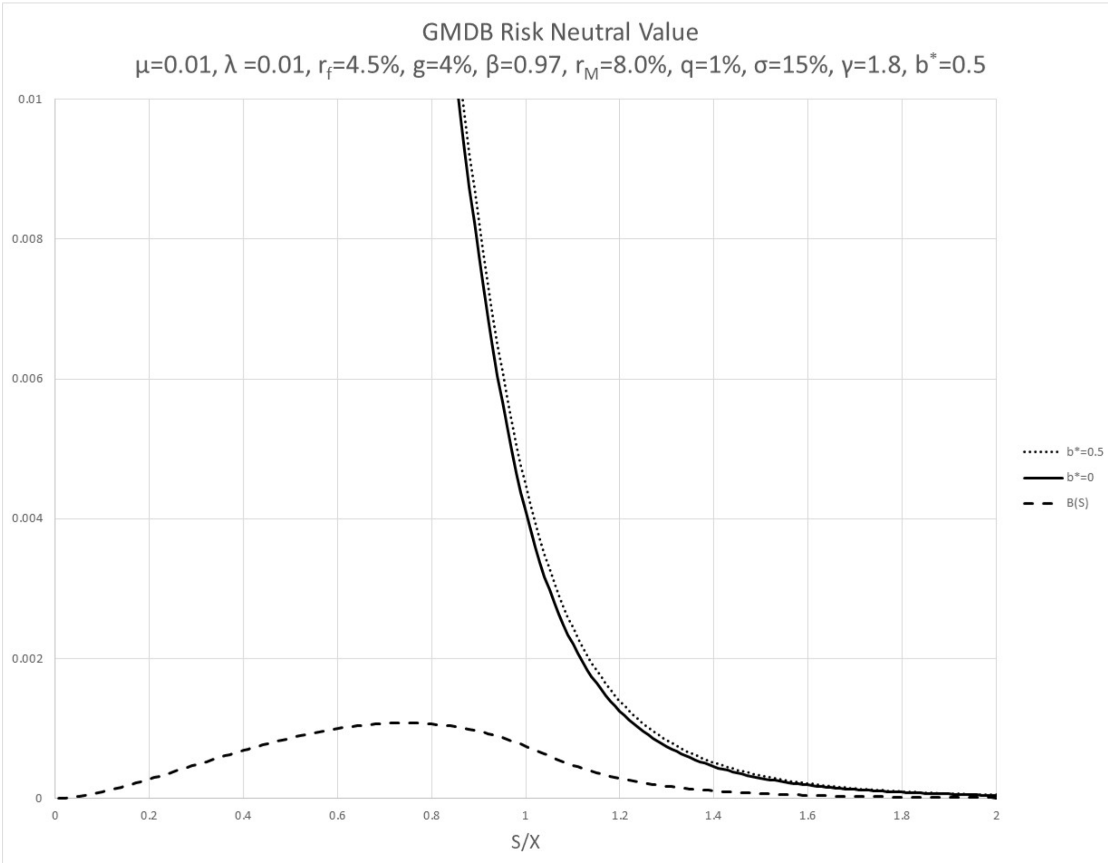

Figure 3
Optimal Allocation for Ratchet GMDB Options

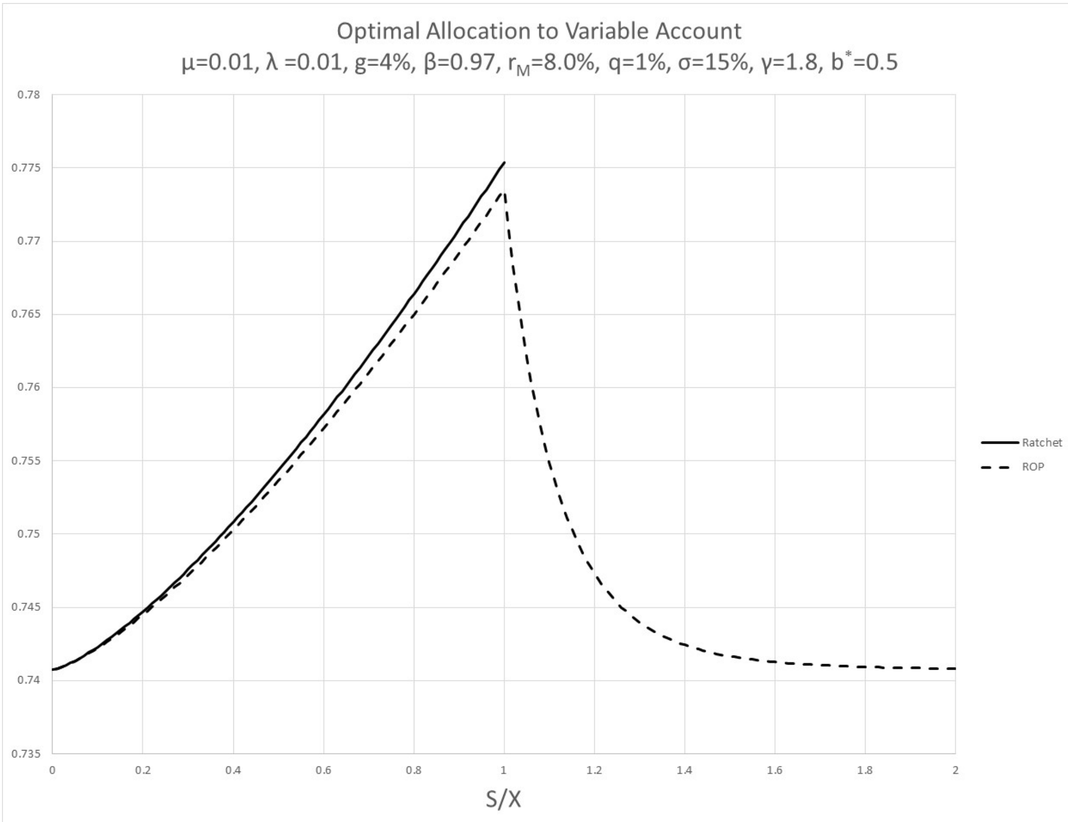

Figure 4
Risk Neutral Value of Ratchet GMDB Option
With and Without Bequest Motive

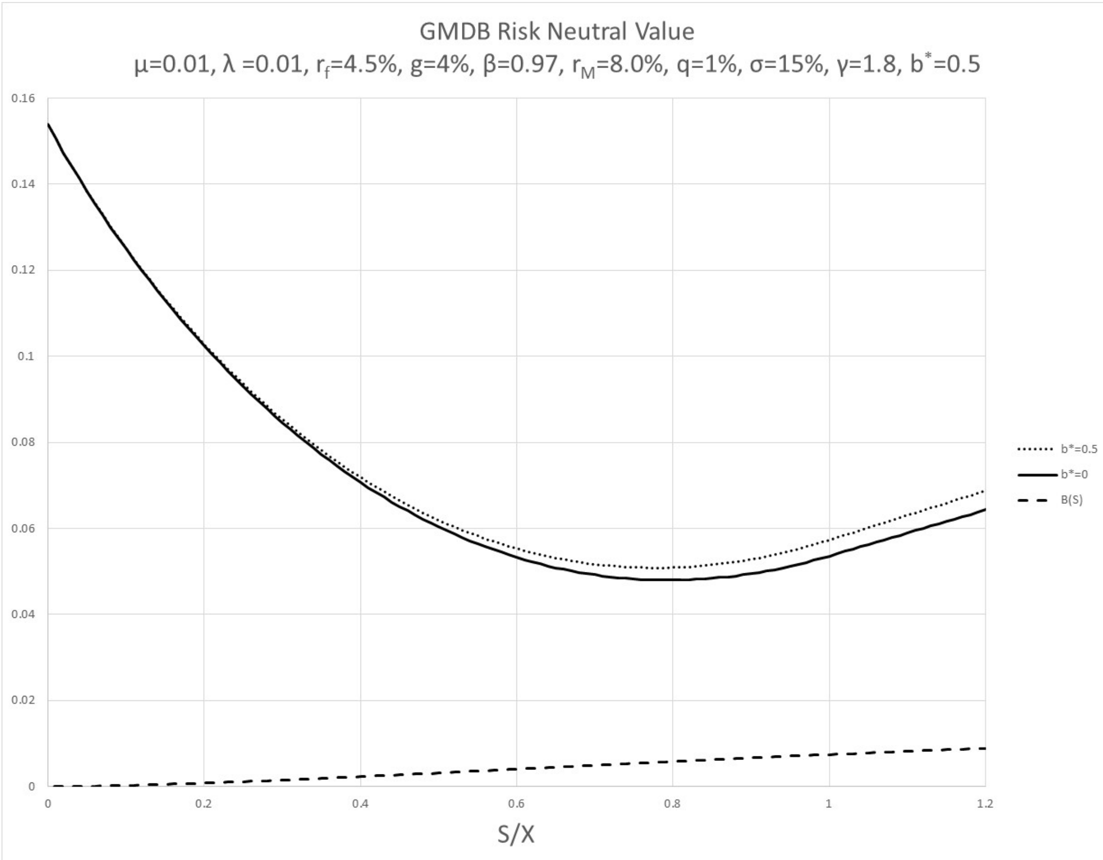

Figure 5
Optimal Withdrawals for a Return of Premium GMDB

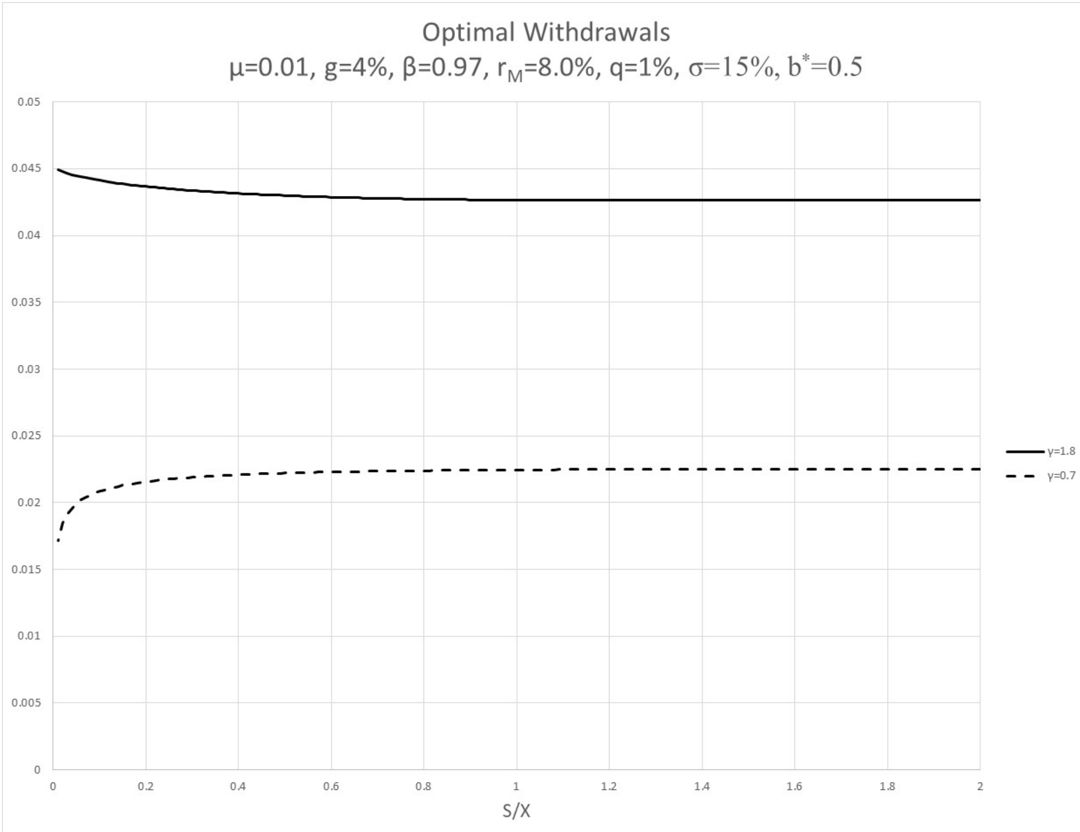

Figure 6
Risk Neutral Value of Return of Premium GMDB Option
With Optimal Withdrawals

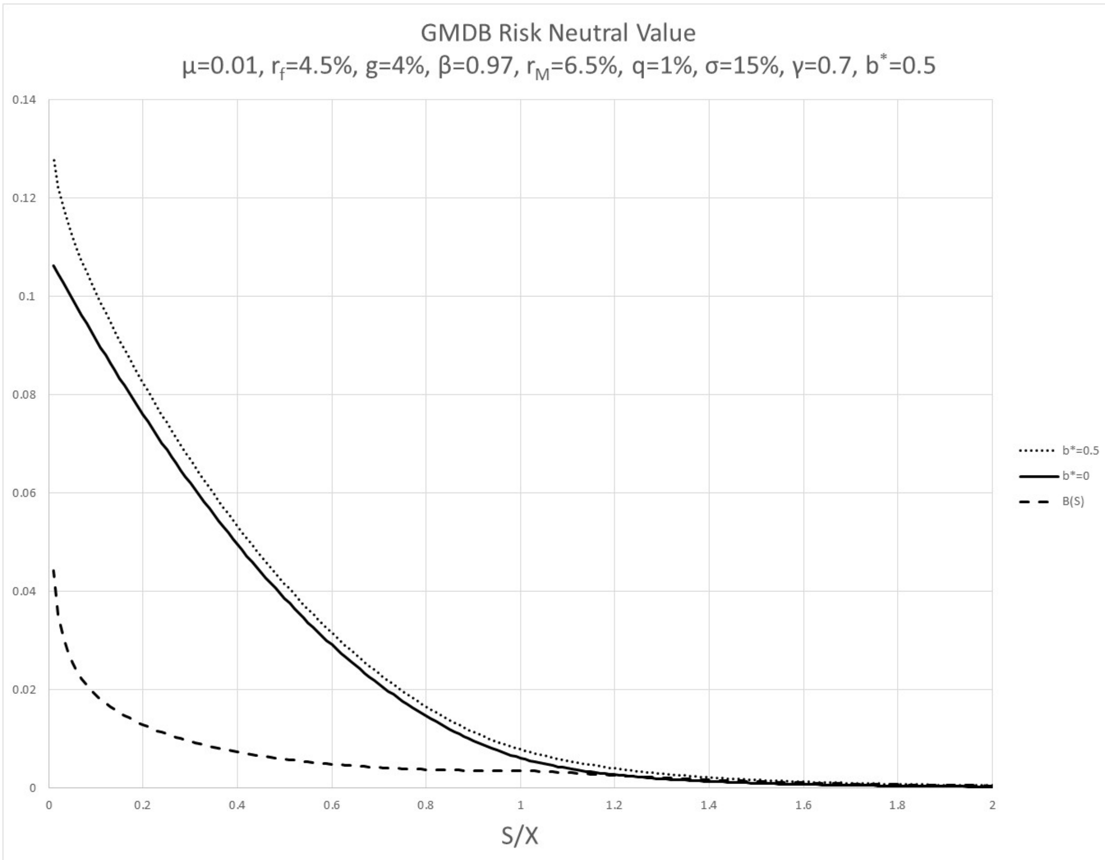

Figure 7
Risk Neutral Value of Return of Premium GMDB Option
With Optimal Withdrawals

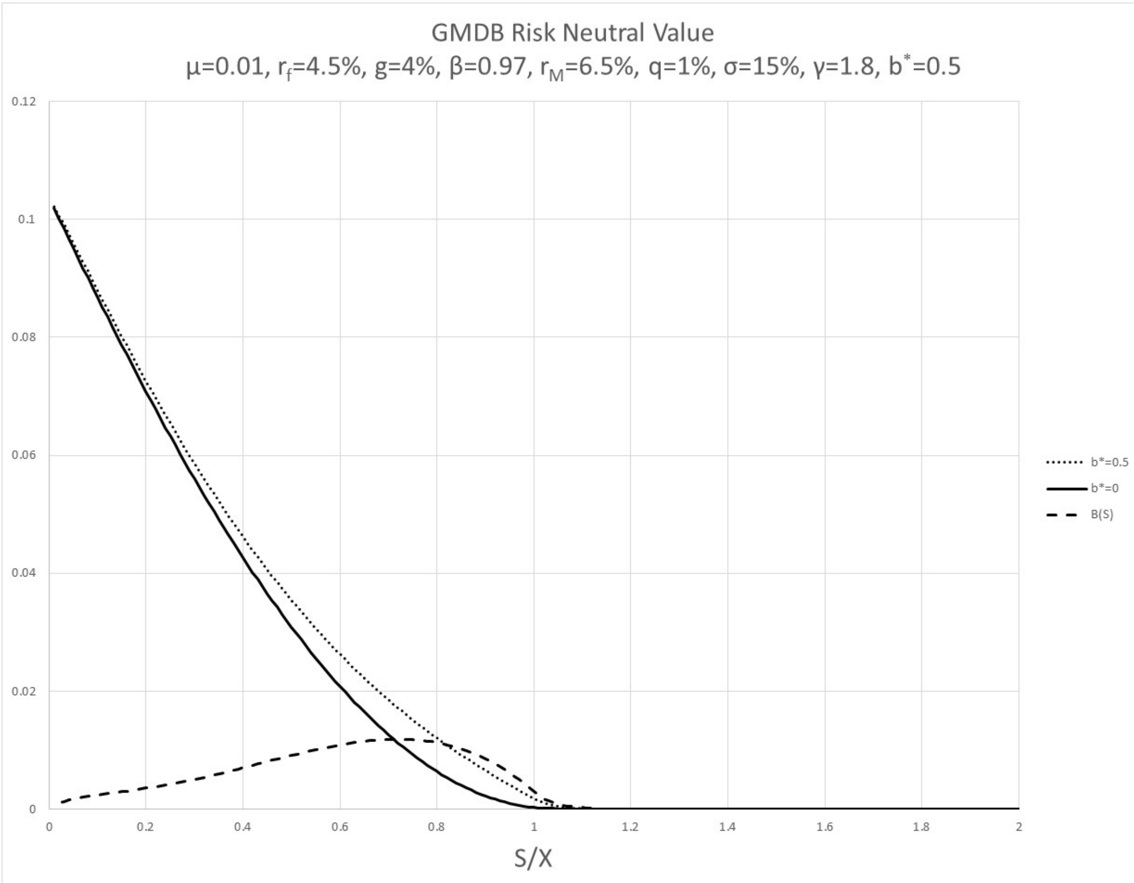

Figure 8
Optimal Allocation for Return of Premium GMDB Options under DeMoivre's Law Mortality

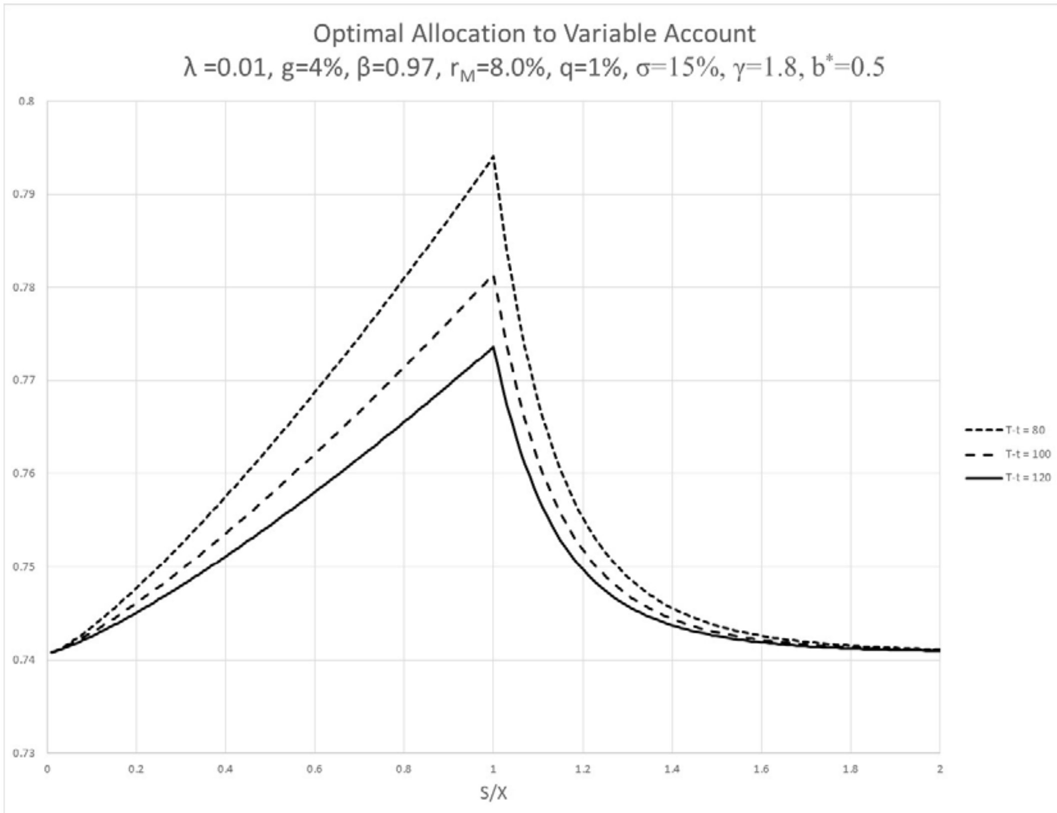

Figure 9
Optimal Allocation for Return of Premium GMDB Options under DeMoivre's Law Mortality

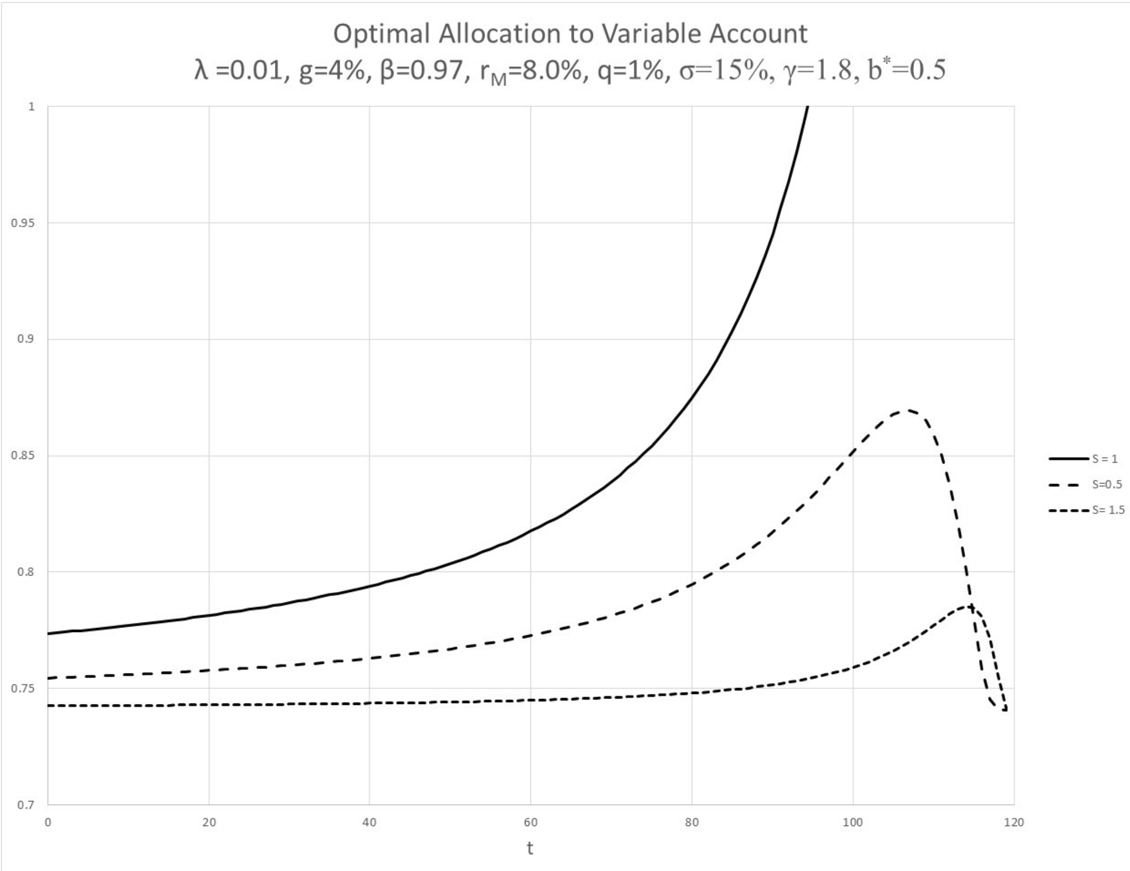

Figure 10
Optimal Allocation for Ratchet GMDB Options under DeMoivre's Law Mortality

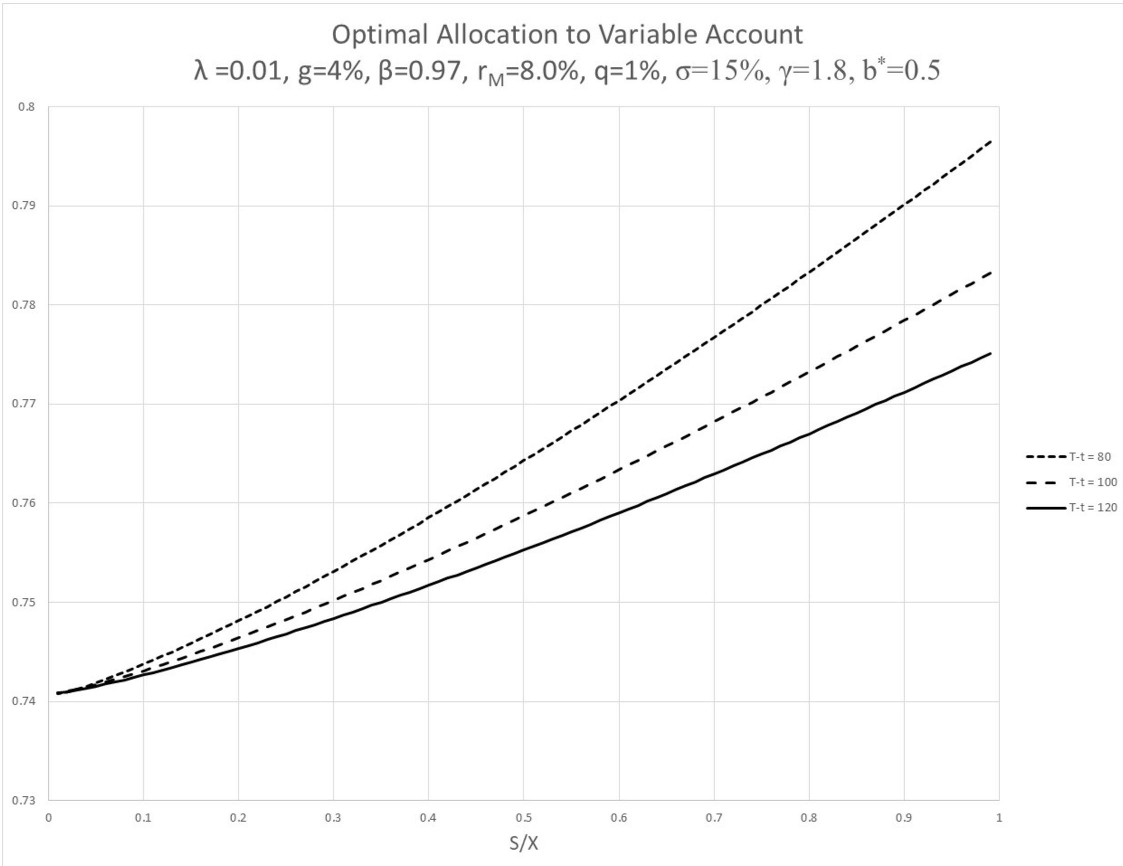

Figure 11
Optimal Allocation for Ratchet GMDB Options under DeMoivre's Law Mortality

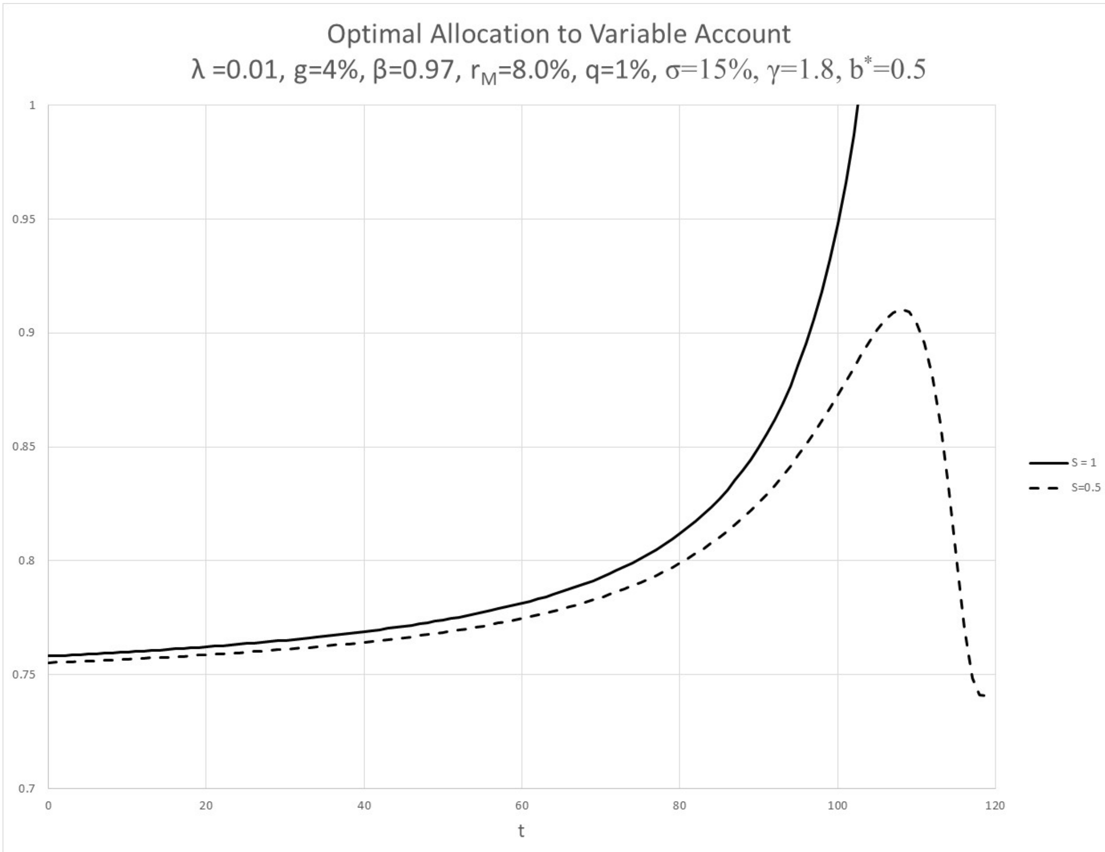

Table 1
Return of Premium At-the-Money GMDB Option Value With and Without Bequest Motive

g	r_f	R_M	q	σ	β	γ	b^*	μ	λ	No Bequest Motive	Bequest Motive	% Increase
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.002360	0.002592	9.82%
0.02	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.007110	0.007455	4.84%
0.04	0.07	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.001220	0.001340	9.86%
0.04	0.05	0.1	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.007908	0.008287	4.79%
0.04	0.05	0.08	0.02	0.15	0.97	1.8	0.5	0.01	0.05	0.000933	0.001061	13.69%
0.04	0.05	0.08	0.01	0.25	0.97	1.8	0.5	0.01	0.05	0.000508	0.000584	14.89%
0.04	0.05	0.08	0.01	0.15	0.99	1.8	0.5	0.01	0.05	0.002360	0.002621	11.04%
0.04	0.05	0.08	0.01	0.15	0.97	3	0.5	0.01	0.05	0.000508	0.000554	8.99%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.2	0.01	0.05	0.002360	0.002453	3.93%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.02	0.05	0.004406	0.005214	18.34%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.1	0.001727	0.001927	11.62%

Table 2
Return of Premium GMDB Option Value With and Without Bequest Motive

g	r_f	R_M	q	σ	β	γ	b	μ	λ	No Bequest Motive	Bequest Motive	% Increase
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.018379	0.019847	7.98%
0.02	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.044250	0.047291	6.87%
0.04	0.07	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.013316	0.014442	8.46%
0.04	0.05	0.1	0.01	0.15	0.97	1.8	0.5	0.01	0.05	0.043482	0.045503	4.65%
0.04	0.05	0.08	0.02	0.15	0.97	1.8	0.5	0.01	0.05	0.009179	0.010115	10.20%
0.04	0.05	0.08	0.01	0.25	0.97	1.8	0.5	0.01	0.05	0.007313	0.008109	10.88%
0.04	0.05	0.08	0.01	0.15	0.99	1.8	0.5	0.01	0.05	0.018379	0.020038	9.02%
0.04	0.05	0.08	0.01	0.15	0.97	3	0.5	0.01	0.05	0.007313	0.007793	6.57%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.2	0.01	0.05	0.018379	0.018966	3.19%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.02	0.05	0.031521	0.036242	14.97%
0.04	0.05	0.08	0.01	0.15	0.97	1.8	0.5	0.01	0.1	0.009779	0.010738	9.81%

References:

Gao, J. and Ulm, E. R. (2012). Optimal Consumption and Allocation in Variable Annuities with Guaranteed Minimum Death Benefits, *Insurance: Mathematics and Economics* 51(3), 586-598.

Gao, J. and Ulm, E. R. (2015). Optimal Allocation and Consumption with Guaranteed Minimum Death Benefits, External Income and Term Life Insurance. *Insurance: Mathematics and Economics* 61, 87-98.

Gerber, H.U., Shiu, E.S.W and Yang, H. (2012), Valuing Equity-Linked Death Benefits and Other Contingent Options: A Discounted Density Approach, *Insurance: Mathematics and Economics* 51:73-92

Gerber, H.U., Shiu, E.S.W and Yang, H. (2013), Valuing Equity-Linked Death Benefits in Jump Diffusion Models, *Insurance: Mathematics and Economics* 53:615-623

Hardy, M. (2003), *Investment Guarantees*, Hoboken, N.J.: John Wiley and Sons.

Merton, R.C. (1969), Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case, *Review of Economics and Statistics* 51:247-257

Milevsky, M.A., and Posner, S.E. (2001), The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds, *Journal of Risk and Insurance* 68(1):93-128

Moenig, T., (2012), Optimal Policyholder Behavior in Personal Savings Products and its Impact on Valuation, *Risk Management and Insurance Dissertations* 28, Georgia State University

Moenig, T. and Bauer, D. (2014), Negative Option Values in Personal Savings Products, Working Paper, Georgia State University

Moenig, T., and Bauer, D. (2015). Revisiting the Risk-Neutral Approach to Optimal Policyholder Behavior: A Study of Withdrawal Guarantees in Variable Annuities. *Review of Finance*, 20(2), 759-794.

Moenig, T., and Zhu, N. (2018). Lapse-and-Reentry in Variable Annuities. *Journal of Risk and Insurance*, 85(4), 911-938.

Pham, Huyen (2009) *Continuous Time Stochastic Control and Optimization with Financial Applications,* Stochastic Modelling and Applied Probability 61, Springer Verlag, Berlin

Steinorth, P., and Mitchell, O. S. (2015). Valuing Variable Annuities with Guaranteed Minimum Lifetime Withdrawal Benefits. *Insurance: Mathematics and Economics*, *64*, 246-258.

Ulm, E.R. (2006). The Effect of the Real Option to Transfer on the Value of Guaranteed Minimum Death Benefits, *Journal of Risk and Insurance* 73(1): 43-69

Ulm, E.R. (2008), Analytic Solution for Return of Premium and Rollup Guaranteed Minimum Death Benefit Options Under Some Simple Mortality Laws, *ASTIN Bulletin* 38(2):543-563.

Ulm, E.R. (2014), Analytic Solution for Ratchet Guaranteed Minimum Death Benefit Options Under a Variety of Mortality Laws, *Insurance: Mathematics and Economics* 58:14-23.

Ulm, E.R. (2019), An Overview of Exact Solution Methods for Guaranteed Minimum Death Benefit Options in Variable Annuities, Working Paper, Victoria University of Wellington.

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