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# The Atkinson Inequality Measure and Inequality Aversion\*

John Creedy<sup>†</sup>

## Abstract

This paper examines the precise way in which the Atkinson inequality measure varies as inequality aversion increases. The aim is to investigate whether precise conditions can be obtained under which a tax reform might be judged to be inequality reducing for one range of aversion parameters, and inequality increasing for another range. A number of elasticities, with respect to inequality aversion, are derived and shown to have convenient interpretations. Specific conditions cannot be produced because the Atkinson measure can take the same value for a range of alternative distributions. Nevertheless, intersecting profiles of Atkinson measures plotted against inequality aversion can arise without the need for pathological assumptions about changes in the income distribution. The analysis shows the need to consider a range of aversion parameters when examining changes to the tax and transfer system. By considering only one or two values, it could be concluded incorrectly that a tax reform is progressive, when a higher degree of inequality aversion would judge a change to be regressive.

**JEL Classification:** H23; H24.

**Keywords:** Atkinson inequality measure; inequality aversion; distributional comparisons.

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# 1 Introduction

The need to introduce value judgements explicitly in the measurement of inequality was stressed by Atkinson (1970) in the important paper introducing his eponymous measure.<sup>1</sup> Influenced by recent work on risk aversion, he famously showed how a single parameter, reflecting relative inequality aversion, can be used in combination with a class of social welfare functions expressed in terms of individual incomes, to obtain an inequality measure defined in terms of the proportional difference between arithmetic mean income and an equally distributed equivalent income. The latter is defined as that income which, if equally distributed (so that each person receives arithmetic mean income), gives rise to the same value of social welfare as the actual distribution. His paper included a table showing how the ranking of a number of countries according to measured inequality can vary substantially as the degree of inequality aversion is increased.

A similar kind of re-ranking can arise when considering two distributions of net (that is, post-tax and transfer) incomes for the same country. Hence, a policy change can be judged to reduce inequality for one range of values of inequality aversion, while it increases inequality for other values. Put another way, if the Atkinson measure is plotted against inequality aversion for two distributions of net income, it is possible for the profiles to intersect. Of course, inequality necessarily increases as inequality aversion increases, and ultimately the Atkinson measure (starting from zero when there is no aversion) approaches unity as aversion becomes ‘infinitely high’. Less weight is progressively attached, by the welfare function, to higher incomes as aversion increases. Hence, intuitively speaking, a re-ranking can occur if one distribution introduces more inequality in the very lowest ranges of net incomes, while reducing inequality among higher incomes.

The question considered here is whether anything more specific can be said about two distributions for which the profiles of inequality intersect as aversion increases. An associated question relates to the rate at which the

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<sup>1</sup>On the influence of this paper, which gave rise to a vast literature, see, for example, Lambert (1993) and Jenkins (2016)

profiles of two distributions converge or diverge. Although properties of the Atkinson measure have been extensively investigated, the precise nature of these variation does not seem to have been examined. Section 2, after briefly providing a reminder of the definition of Atkinson's measure, considers the general case. A context in which no intersections can arise is explained in Section 3. Brief conclusions are in Section 4.

## 2 Atkinson's Measure

The Atkinson measure of inequality,  $A$ , of the distribution,  $y_1, \dots, y_n$ , is expressed as:

$$A = 1 - \frac{y_e}{\bar{y}} \quad (1)$$

where  $\bar{y}$  is the arithmetic mean income, and  $y_e$  is the equally distributed equivalent income, defined as:

$$y_e = \left( \frac{1}{n} \sum_{i=1}^n y_i^{1-\varepsilon} \right)^{1/(1-\varepsilon)} \quad (2)$$

The parameter,  $\varepsilon$ , is the degree of relative inequality aversion, with  $\varepsilon \geq 0$  and  $\varepsilon \neq 1$ . When  $\varepsilon = 1$ , the equally distributed equivalent is geometric mean income. Equation (2) is based on the associated welfare function,  $W = \frac{1}{n} \sum_{i=1}^n y_i^{1-\varepsilon} / (1-\varepsilon)$ , which is Paretean, individualistic, additive, and is concerned with relative rather than absolute inequality.

Measured inequality increases with  $\varepsilon$ , and, as Atkinson stressed, the ranking of two distributions can change as  $\varepsilon$  changes. However, inequality typically becomes almost unchanged (approaching unity) as  $\varepsilon$  increases beyond about 5, which virtually reflects extreme aversion. Hence, it is possible for a change in a tax and transfer system to be judged as inequality increasing or decreasing, depending on the degree of relative inequality aversion. To examine whether precise conditions can be established under which the ranking of two distributions changes, and the rate at which inequality of two distributions converges or diverges, subsection 2.1 derives several elasticities. Numerical examples are given in subsection 2.2.

## 2.1 Variations in Alternative Measures and Elasticities

Direct differentiation of  $A$  with respect to  $\varepsilon$  is obviously not straightforward. Consider instead, looking at equality,  $E = 1 - A$ , rather than inequality, so that:

$$E = \frac{y_e}{\bar{y}} \quad (3)$$

and:

$$\log E = \log y_e - \log \bar{y} \quad (4)$$

Letting  $a = 1 - \varepsilon$ , differentiation of (4) with respect to  $a$  gives:

$$\frac{d \log E}{da} = \frac{d \log y_e}{da} \quad (5)$$

The change in log-equality as  $a$  increases is therefore simply the change in the logarithm of equally distributed equivalent income. Multiplying both sides of (5) by  $a$ , and using the notation,  $\eta_{z,x}$ , to denote the elasticity of  $z$  with respect to  $x$ , gives (since  $d \log x = dx/x$ ):

$$\eta_{E,a} = \eta_{y_e,a} \quad (6)$$

From (2), which becomes  $\log y_e = \frac{1}{a} \log \left( \frac{1}{n} \sum_{i=1}^n y_i^a \right)$ :

$$\frac{d \log y_e}{da} = -\frac{1}{a^2} \log \left( \frac{1}{n} \sum_{i=1}^n y_i^a \right) + \frac{1}{a} \frac{d \log \left( \frac{1}{n} \sum_{i=1}^n y_i^a \right)}{da} \quad (7)$$

This can be expressed more succinctly as:

$$\eta_{\log y_e, a} = -1 + \eta_{\log \left( \frac{1}{n} \sum_{i=1}^n y_i^a \right), a} \quad (8)$$

Consider, then,  $d \log \left( \frac{1}{n} \sum_{i=1}^n y_i^{1-\varepsilon} \right) / da$ , and again using  $d \log x = dx/x$ :

$$\frac{d \log \left( \frac{1}{n} \sum_{i=1}^n y_i^a \right)}{da} = \frac{1}{\sum_{i=1}^n y_i^a} \frac{d \left( \sum_{i=1}^n y_i^a \right)}{da} \quad (9)$$

Now consider  $\frac{d \left( \sum_{i=1}^n y_i^a \right)}{da}$ . In general, for constant,  $b$ , and variable,  $x$ :

$$\frac{d}{dx} (b^x) = b^x \log b \quad (10)$$

Hence:

$$\frac{d(\sum_{i=1}^n y_i^a)}{da} = \sum_{i=1}^n y_i^a \log y_i \quad (11)$$

Substituting this result in (9) and writing in elasticity form gives:

$$\eta_{\log(\frac{1}{n} \sum_{i=1}^n y_i^a), a} = a \frac{\sum_{i=1}^n (y_i^a / \sum_{i=1}^n y_i^a) \log y_i}{\log(\frac{1}{n} \sum_{i=1}^n y_i^a)} \quad (12)$$

In general, elasticities of a variable and the logarithm of that variable are related by the simple relationship:

$$\eta_{x,z} = (\log x) \eta_{\log x, z} \quad (13)$$

Hence:

$$\eta_{y_e, a} = (\log y_e) \eta_{\log y_e, a} \quad (14)$$

$$= (\log y_e) \left\{ -1 + a \frac{\sum_{i=1}^n (y_i^a / \sum_{i=1}^n y_i^a) \log y_i}{\log(\frac{1}{n} \sum_{i=1}^n y_i^a)} \right\} \quad (15)$$

and using the fact that  $y_e = E\bar{y}$ :

$$\eta_{E, a} = \eta_{y_e, a} = \sum_{i=1}^n \left( \frac{y_i^a}{\sum_{i=1}^n y_i^a} \right) \log y_i - (\log \bar{y} E) \quad (16)$$

This elasticity has a convenient interpretation. The proportional change in equality, resulting from a proportional change in inequality aversion, is therefore the difference between a weighted average of log-income and the logarithm of ‘equality adjusted’ arithmetic mean income.

Furthermore, (16) can be converted into an elasticity of  $A$  with respect to  $\varepsilon$ , as follows:

$$\eta_{A, \varepsilon} = \eta_{E, a} \left( \frac{\varepsilon}{1 - \varepsilon} \right) \left( \frac{1 - A}{A} \right) \quad (17)$$

Similarly, the elasticity of  $E$  with respect to  $\varepsilon$  is related to  $\eta_{E, a}$  using:

$$\eta_{E, \varepsilon} = -\eta_{E, a} \left( \frac{\varepsilon}{1 - \varepsilon} \right) \quad (18)$$

Consider the special case of  $a = 0$ , corresponding to  $\varepsilon = 1$ . In this case  $\sum_{i=1}^n \left( \frac{y_i^a}{\sum_{i=1}^n y_i^a} \right) \log y_i = \frac{1}{n} \sum_{i=1}^n \log y_i$ , which is the logarithm of geometric

mean income. Similarly,  $E$  is the ratio of the geometric mean income to arithmetic mean, so that  $\log \bar{y}E$  is also equal to the logarithm of geometric mean. Hence  $\eta_{E,a=0} = 0$  when  $\varepsilon = 1$ . Alternatively, when  $a = 1$ , corresponding to  $\varepsilon = 0$ , substitution into (16) gives:

$$\eta_{E,a=1} = \frac{1}{n} \sum_{i=1}^n \left( \frac{y_i}{\bar{y}} \right) \log y_i - (\log \bar{y}) \quad (19)$$

This result shows that  $\eta_{E,a=1}$  is the difference between a share-weighted mean log-income and the logarithm of arithmetic mean income. This is positive, so that  $\eta_{E,a}$  begins positive for low  $\varepsilon$  and becomes negative for  $\varepsilon > 1$ . This can also be seen by returning to the simple relationship between elasticities, whereby the term,  $\eta_{E,a} = \eta_{E,1-\varepsilon}$  is also expressed in terms of  $\eta_{E,\varepsilon}$  as:

$$\eta_{E,a} = - \left( \frac{1-\varepsilon}{\varepsilon} \right) \eta_{E,\varepsilon} \quad (20)$$

Clearly  $\eta_{E,\varepsilon}$  is negative for  $\varepsilon > 0$ : inequality is necessarily higher as inequality aversion increases. Hence  $\eta_{E,a} > 0$  when  $\varepsilon < 0$ , and  $\eta_{E,a} < 0$  when  $\varepsilon > 1$ .

The typical shapes of the various profiles can be illustrated by taking a simple numerical example. Suppose there are just 8 individuals, with incomes in ascending order given by: 5, 10, 20, 50, 100, 300, 500, 1000. Figure 1 shows how  $A$  and  $E$  vary as  $\varepsilon$  is increased. Figure 2 plots the three elasticities,  $\eta_{1-A,1-\varepsilon}$ ,  $\eta_{A,\varepsilon}$  and  $\eta_{1-A,\varepsilon}$ , as  $\varepsilon$  varies. The more extensive distributions found in practice will nevertheless give rise to similarly shaped smooth profiles.

## 2.2 Comparisons Between Two Distributions

Having derived a number of elasticities and considered their shapes, the question is then whether this can be used to say anything specific about the properties of the distributions for which the inequality ranking changes as inequality aversion increases. A fundamental problem immediately arises because the Atkinson measure, just like the famous Gini inequality measure, can take the same value for a range of quite different distributions: this feature is explored in detail in Creedy (2017). Hence in general it does not seem possible to specify a particular type of change. A similar problem arises in



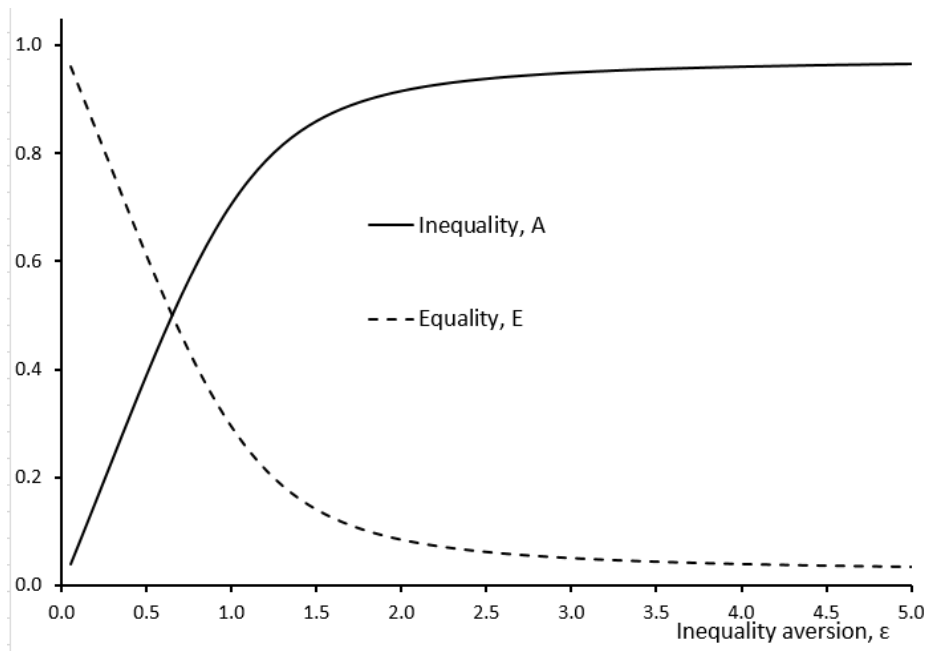


Figure 1: Variation in Inequality and Equality with Inequality Aversion

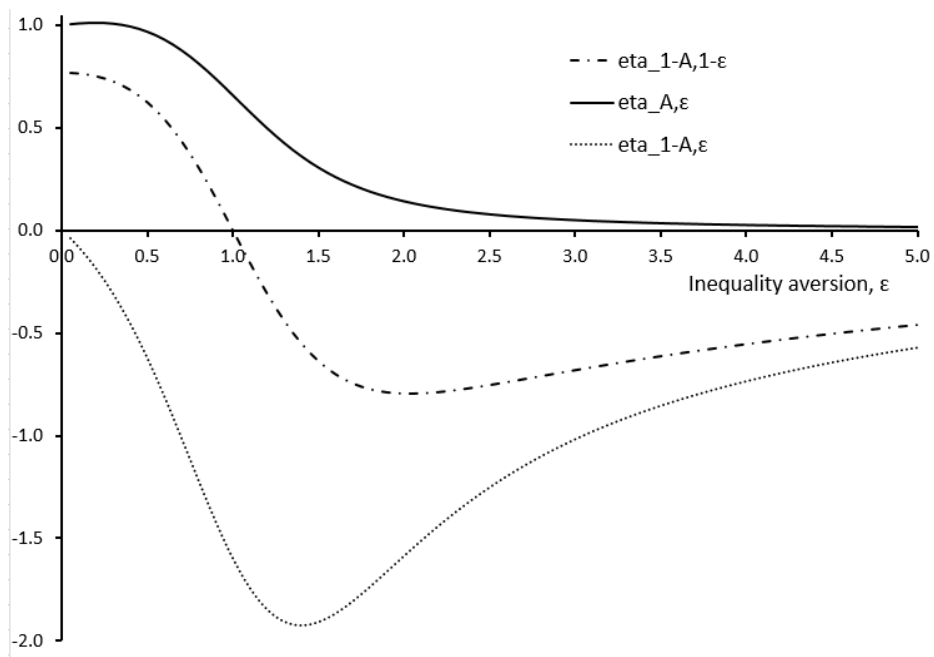


Figure 2: Variation in Elasticities With Inequality Aversion

attempting to determine a ‘pivotal income’ for the Atkinson measure: this concept may be regarded as a dividing line between rich and poor, in that an increase in any income below the pivotal income reduces inequality, for a given degree of inequality aversion. The pivotal income can be expressed in terms of  $\bar{y}$ ,  $A$  and  $\varepsilon$ , as shown by Creedy (2016), who gives a special case of the more general results of Lambert and Lanza (2006). Yet this requires full information about the precise income distribution.

Of course, it would be possible to consider, say, minimal changes involving changes in, say, just two incomes, with the rest of the distribution held constant. A crossing point could be determined by solving the resulting nonlinear equation. But this involves full knowledge of the remaining distribution (since a sum of powers is involved), and would not necessarily give a unique solution.

In the case of the simple distribution used in the previous subsection, a change involving an increase in one lower income (say increasing 20 to 25, or raising 50 to 60), but not the lowest income, does reduce inequality for lower values of inequality aversion. But for higher degrees of aversion – 3.9 and 2.25 respectively for the two examples – the reduced weight given to those lower income implies that inequality increases. A reduction in the bottom income must necessarily reduce inequality for all values of  $\varepsilon$ .

An intersection of the profiles of  $A$  against  $\varepsilon$ , for lower values of  $\varepsilon$ , can also be achieved by changing two incomes in the bottom tail of the distribution. Thus reducing 5 to 4, and at the same time raising 10 to 15, reduces inequality for  $\varepsilon < 1.55$ , after which  $A$  is higher than in the first distribution. Figure 3 illustrates the case for a second distribution in which the lowest three incomes are changed to 4, 18, and 25, while the remaining five values are unchanged.

The analysis has shown that full information is needed about the income distribution if specific conditions for intersecting profiles of  $A$  against  $\varepsilon$  are to be determined. A special case clearly arises if the distribution can be described by a particular functional form involving a small number of parameters. An example is discussed in the following section.

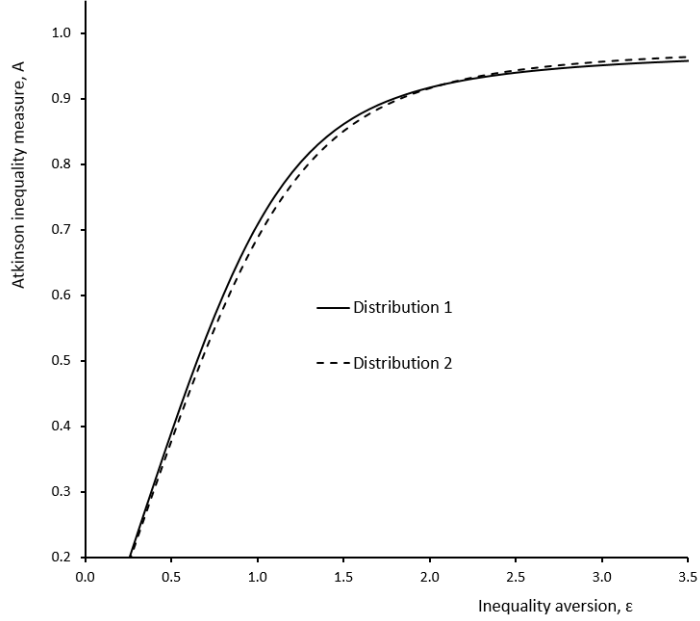


Figure 3: Variation in Atkinson Measure with Epsilon for Two Distributions

### 3 The Lognormal Distribution

Suppose  $y$  is lognormally distributed as  $\Lambda(y|\mu, \sigma^2)$ , where  $\mu$  and  $\sigma^2$  are respectively the mean and variance of  $\log y$ : for details, see Aitchison and Brown (1957). From the moment generating function, the arithmetic mean is:

$$\bar{y} = \exp\left(\mu + \frac{\sigma^2}{2}\right) \quad (21)$$

and the power mean,  $y_\varepsilon$ , is given by:

$$y_\varepsilon = \left[ \exp\left\{ (1 - \varepsilon)\mu + \frac{\sigma^2(1 - \varepsilon)^2}{2} \right\} \right]^{\frac{1}{1 - \varepsilon}} \quad (22)$$

Taking logs and subtracting, gives:

$$\log \frac{y_\varepsilon}{\bar{y}} = -\frac{\varepsilon\sigma^2}{2} \quad (23)$$

Hence, Atkinson's measure becomes:

$$A = 1 - \exp\left(-\frac{\varepsilon\sigma^2}{2}\right) \quad (24)$$

Differentiating with respect to  $\varepsilon$  gives:

$$\frac{dA}{d\varepsilon} = \frac{\sigma^2}{2} \exp\left(-\frac{\varepsilon\sigma^2}{2}\right) \quad (25)$$

The elasticity,  $\eta_{A,\varepsilon}$ , is thus:

$$\eta_{A,\varepsilon} = \frac{\varepsilon\sigma^2/2}{\exp(\varepsilon\sigma^2/2) - 1} \quad (26)$$

The result in (24) shows immediately that, for the lognormal case, a distributional change – involving a change in  $\sigma^2$  – must lead to a consistent upward or downward movement of the profile of  $A$  against  $\varepsilon$ . Therefore, intersections cannot occur. An example of inequality and equality profiles is given in Figure 4 for  $\sigma^2 = 0.4$ .

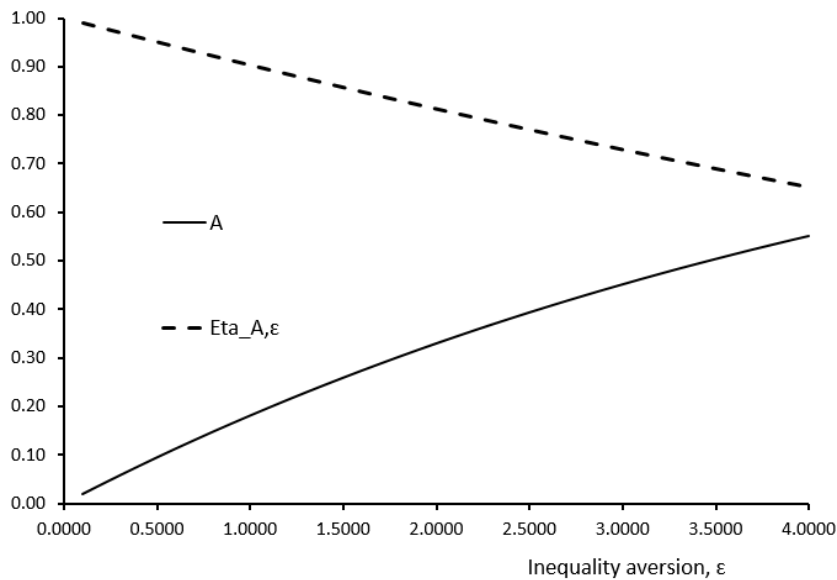


Figure 4: Variations in Atkinson Measure and Elasticity with Respect to Epsilon

This property suggests that there is a need to be concerned about intersecting profiles, when comparing two distributions, to the extent that they deviate from lognormality. This distribution is known to provide a reasonable approximation over the complete range of incomes for many empirical

distributions. However, particularly when examining net incomes, distributions can have spikes associated with thresholds relating to means-tested benefits, as well as those which may be associated with income tax thresholds.<sup>2</sup> For example, in the context of the New Zealand distribution of net incomes, there is a large spike in the lower tail associated with New Zealand Superannuation.

## 4 Conclusions

This paper has examined the precise way in which the Atkinson inequality measure varies as the degree of inequality aversion increases. The motivation for the analysis was the desire to see if particular conditions could be obtained under which, say, a reform to the direct tax and transfer system might be judged to be inequality reducing for one range of aversion parameters, and inequality increasing for another range. A number of elasticities, with respect to inequality aversion, were derived and were shown to have convenient interpretations. Yet, the fact that the Atkinson measure can have the same value for two quite different distributions means also that specific conditions cannot be produced.

Nevertheless, it is seen that intersections of profiles of the Atkinson measures against the inequality aversion parameter can arise without the need for pathological assumptions about income changes. Many distributional changes, involving higher inequality in the lower-income ranges of the distribution, are generally capable of producing intersections. Thus, while specific changes cannot be determined, the analysis shows the need to consider a range of aversion parameters when examining an actual or proposed change to the tax and transfer system. By considering only one or two values, it could be concluded incorrectly that a tax reform is progressive, when someone with a high degree of inequality aversion, and thus a strong interest in achieving more redistribution, would judge a change to be regressive.

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<sup>2</sup>The lognormal form is particularly useful when constructing a range of economic models where the distribution is merely one component and it is necessary to be able to describe the distribution succinctly, say for purposes of aggregation. In such cases, deviations such as the spikes discussed here may not be important.

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