Transition paths for Bewley-Huggett-Aiyagari models: Comparison of some solution algorithms

Robert Kirkby

SEF WORKING PAPER 1/2017

The Working Paper series is published by the School of Economics and Finance to provide staff and research students the opportunity to expose their research to a wider audience. The opinions and views expressed in these papers are not necessarily reflective of views held by the school. Comments and feedback from readers would be welcomed by the author(s).

Further enquiries to:

The Administrator
School of Economics and Finance
Victoria University of Wellington
P O Box 600
Wellington 6140
New Zealand

Phone: +64 4 463 5353
Email: alice.fong@vuw.ac.nz

Working Paper 1/2017

ISSN 2230-259X (Print)

ISSN 2230-2603 (Online)

Transition Paths for Bewley-Huggett-Aiyagari Models: Comparison of some Solution Algorithms

Robert Kirkby*

January 13, 2017

Abstract

We analyse general equilibrium transition paths for Bewley-Huggett-Aiyagari models: general equilibrium models with heterogeneous agents, incomplete markets, and idiosyncratic but no aggregate uncertainty. We first provide precise definitions of the theoretical problem to be solved. We then consider a variety of algorithms for computation of the finite horizon general equilibrium transition paths. These algorithms can solve for unannounced one-off changes, pre-announced one-off changes, or even any finite series of one-off changes (such as announcing today a series of tax increases to be rolled out over the next few years). The algorithms are all based on shooting-algorithms but differ in how they update the price paths during convergence. Evaluation of the algorithms in terms of both robustness and runtime is performed by applying them to looking at capital tax reforms in the model of Aiyagari (1994) and its extension to endogenous labour. Simulation results show that the literature standard of simply using a fixed factor to update the entire path performs both fast and robustly; and that a weight of 0.9 on the old path is typically the best by these criterion.

Keywords: Bewley-Huggett-Aiyagari models, Numerical methods, Transition Path, General Equilibrium.

JEL Classification: E00; C68; C63; C62

*Kirkby: Victoria University of Wellington. Please address all correspondence about this article to Robert Kirkby at <robertdkirkby@gmail.com>, robertdkirkby.com. I acknowledge the useful comments of seminar participants at Computational Economics and Finance (2016).

Contents

1 Introduction **1**

2 Model Environment and Definitions **4**
2.1 Stationary Competitive Equilibrium 4
2.2 Transition Path 5
2.3 Finite Horizon Transition Paths 7

3 Algorithms for Computation **8**

4 Results from Application of Algorithms **11**

5 Conclusion **14**

A The Specific Model **17**

B Extension of Application to Endogenous Labour supply **19**

1 Introduction

Quantitative macroeconomic models are often used to perform policy evaluations. The typical study proceeds as follows. First calibrate/estimate the stationary equilibrium of the model based on current policies and the economy; the 'before'. Second, change the policy and find the resulting stationary equilibrium of the model; the 'after'. Increasingly a third step, considering the *transition* from the 'before' to the 'after' is considered. Bewley-Huggett-Aiyagari models — general equilibrium models with heterogeneous agents, incomplete markets, and idiosyncratic but no aggregate uncertainty — in which such analysis is often performed, especially for fiscal issues where both distributional and efficiency concerns are important. The first two steps, the before and after, are grounded in a theoretical literature. The third step is done computationally and little theory exists. This paper systematically compares a number of variations to algorithms for the computation of this third step of finding a general equilibrium transition path.

Why should we consider transition paths? For policy evaluation two main views of their importance might be taken, based in their theoretical and empirical importance respectively.

The theoretical importance of transition paths is well illustrated by Lucas (1990) who looks at the potential welfare gains from reducing capital taxes in a representative agent model. For a model with an infinitely-lived representative agent and endogenous savings it is well known result that the optimal (stationary competitive equilibrium) capital tax rate is zero Judd (1985); Chamley (1986). Lucas shows that a simple steady-state comparison of a capital tax rate calibrated to match US tax revenue data and a capital tax rate of zero suggests a very large potential welfare gain from moving to a capital tax rate of zero. Lucas then shows however that once the transition is taken into account the potential welfare gain is notably smaller; intuitively, to take advantage of the new lower capital tax rate households must save more now, reducing present consumption and lowering the present value of the welfare gains.

The empirical importance of transition paths is well illustrated by Auerbach and Slemrod (1997) who summarize the findings of the literature on the 1986 Tax Reform Act in the USA. They find that one of the effects of the reform most evident in the data was that people shifted asset sales in time to take advantage of the difference between current tax rates and announced future tax rates. This kind of effect will obviously be completely absent of models based on steady-state comparisons. Thus to understand more fully the impacts of a tax reform on government revenues we must model the transition during which such effects can occour.

Unfortunately little is known analytically about the properties of transition paths, in Bewley-Huggett-Aiyagari (BHA) models or most other dynamic Macroeconomic models. Kehoe and Levine (1985) look at the related issue of convergence to stationary equilibria in overlapping-generation models, albeit from the perspective of whether transition paths can be used to pin down existence and uniqueness of the stationary equilibria themselves, rather than focusing on the properties of the transition path. Acikgov (2015) looking at capital taxation in a BHA model finds that *asymptotic optimality* is different from just looking at the optimal stationary equilibrium, and that while it is possible to characterize the asymptotically optimal tax rate without reference to the transition path, there is no avoiding calculation of the transition path in understanding how to get to the asymptotically optimal solution.

While the previous literature contains little theory relating to transition paths it does contain a number quantitative papers which compute transition paths. Conesa and Krueger (1999) compute a transition path in an overlapping-generation model of social security reform, an exercise similar to but slightly different from that we consider, but in which similar shooting-algorithms are applied. Fehr and Kindermann (2015) look at transition paths to re-evaluate the steady-state optimal taxation results of Conesa, Kitao, and Krueger (2009); also in an OLG model. Bakis, Kaymak, and Poschke (2015) look at optimal progressivity of income taxes in model of Aiyagari (1994) fully accounting for the transition path. Kitao (2008) looks at capital tax reforms in a model with entrepreneurship and shows that the transition path also plays an important role in determining which agents would support a given reform. Essentially all of these papers solve the transition path using a shooting-algorithm to iterate on the path of prices and updating the path of prices using a constant weight.

The 'look' of impulse reponse functions and transition paths suggest that most of the movement in prices along transition paths occours at the beginning of the transition path. This suggests looking at algorithms based on a weighting that does 'more' updating of the beginning of the price path and less updating of the tail of the price path. We compare a number of weighting schemes for shooting algorithms that formalize this idea based on exponentially decreasing weights, on opening-windows for updates, and on a combination of the two. We give pseudocode for how to compute transition paths and describe the various algorithms/weightings considered in this paper. To compare the performance of the algorithms/weightings we look at runtime, robustness, and accuracy. We make this comparison looking at transition paths for capital tax reforms in the model Aiyagari (1994); or more precisely its extension to include capital taxation.[1] We consider a number of capital tax reforms: a one-off unannounced reduction in capital tax rates, a one-off pre-announced reduction in capital tax rates, a series of pre-announced staggered cuts in capital tax rates, and 'no-change' (and the analogous tax increases).[2] In Appendix B we further consider the extension of the model to include endogenous labour supply (Pijoan-Mas, 2006). In all cases government revenue is simply spent on a government consumption good which provides additional utility; this makes it irrelevant for our current interest of calculating transition paths. We find the

[1]This model is slightly different from the modelling of a capital tax in Aiyagari (1995) where home production is also introduced.

[2]We choose to look at capital tax reforms simply as there exists a substantial literature on this issue and the important role of transition paths relation to this (Chamley, 1986; Judd, 1985; Lucas, 1990; Aiyagari, 1995; Domeij and Heathcote, 2004; Conesa, Kitao, and Krueger, 2009; Floden, 2009; Acikgov, 2015). It therefore seems an appropriate benchmark for modelling of general equilibrium transition paths.

current standard approach of using a constant weight is the top performer in terms of combining speed and robustness across a range of transition path problems, specifically when using a weight of 0.9 for the old price path.

The algorithms all provide nonlinear solutions for the transition path. Atolia, Chatterjee, and Turnovsky (2010) compare linear and nonlinear solutions to the transition path in a deterministic neoclassical growth model (the Ramsey model). They find that when the transition speeds are slow or involve large deviations away from the steady state, the linear solution methods, based on approximation around the steady state, often result in substantial numerical errors.[3] This was especially so for welfare evaluation and the initial impacts. Errors arising in using linear methods to solve transition paths on rare occasion even lead to qualitatively wrong responses in the early stages of the transition. Given that in heterogeneous agent BHA models the distribution of agents means that there are always some agents far from the steady state, combined with transition paths that typically take a number of periods to converge, it is highly likely that the use of nonlinear methods to solve for transition paths is quantitatively, and possibly even qualitatively, important.

Allowing for a sequence of surprise/unannounced changes will not be considered here but would involve repeatedly solving, in a nested manner, the transition path problem analysed here. Thus any algorithm capable of solving the transition path problem we analyse could be applied in a nested manner to solve a sequence of unannounced changes, although this would likely be an inefficient approach. Alternatively, a series of surprise changes might be better modelled as a series of shocks to the economy, drawn in a standard manner from a probability distribution that reflects possible future reforms, and so it would be a case of just modelling them as a standard exogenous shock and calculating impulse response functions to that series of shocks. This would change the model from a Bewley-Huggett-Aiyagari heterogeneous agent model to a Krusell-Smith model with aggregate uncertainty and so require the standard algorithms used to compute such models (Den Haan, Julliard, and Judd, 2010). Whether a given problem is better modelled as a transition between steady-states or as a shock that changes the exogenous state is a philosophical/empirical issue and best left to the modellers judgment; or ideally, one might model both and see which performs best empirically.

In Section 2 we outline the framework of a general Bewley-Huggett-Aiyagari model. We give a standard definition of a stationary competitive equilibrium as a means to introduce the notation, and then definitions for a general equilibrium transition path and an 'approximate general equilibrium' finite horizon transition path. Section 3 looks at the performance of a variety of algorithms for computing the transition path inspired by a mix a intuition and the transition path theory we developed. Judging the performance of algorithms on both robustness and speed we find that the current literature standard of using a current weight is the best performer across a variety of transition path problems, specifically when a weight of 0.9 is given to the old price path.

[3]For rapid transitions and those that remain near the steady state the errors from linearization were negligible.

2 Model Environment and Definitions

We now describe a general Bewley-Huggett-Aiyagari (BHA) model. We first describe the model environment and provide the definition of a stationary competitive equilibrium. While this definition is standard we use it to introduce notation to index the stationary competitive equilibrium by the parameters and general equilibrium price vector, the later as there are possibly multiple competitive equilibria and it is important for the theory and algorithms that we are explicit about which equilibria.

We then define the infinite horizon transition path problem. The infinite horizon transition path represents the true transition path problem. In practice however computation of transition paths is always implemented as a finite horizon problem. We thus also provide a definition for a finite horizon ϵ-transtition path which the computational algorithms will solve, justifiable following the assumption made throughout the literature that if these finite horizon transition paths converge then it is to an infinite horizon transition path.

2.1 Stationary Competitive Equilibrium

This section gives a formal description of the class of models being considered here; namely general equilibrium heterogeneous agent models with incomplete markets and idiosyncratic, but no aggregate, uncertainty. Notation is loosely based on Stokey, Lucas, and Prescott (1989, henceforth SLP); loosely as SLP do not treat heterogeneous agent models of this type.

The models are those which can be expressed as follows: Let $X \subseteq \mathbb{R}^l$ be the endogenous state variable, $Y = Y_1 \times Y_2 \subseteq X \times \mathbb{R}^c$ be the choice variable, and $Z \subseteq \mathbb{R}^k$ be the exogenous state variable. Let $\Theta \subseteq \mathbb{R}^q$ be a parameter space, and $\theta \in \Theta$ a parameter vector.[4] The state of a agent is then a pair (x, z). A value function maps $V_{p,\theta} : X \times Z \to \mathbb{R}$. A policy function maps $g_{p,\theta} : X \times Z \to Y$. Let $S = X \times Z$, and let $\mathcal{S}$ be it's Borel σ-field. The measure of agents $\mu_{p,\theta}$ is a probability distribution over $(S, \mathcal{S})$. The return function maps $F_{p,\theta} : X \times Y \times Z \to \mathbb{R}$, and the discount factor is $0 < \beta < 1$ (β is considered to be an element of θ).

Aggregate variables are $A \in \mathbb{A} \subseteq \mathbb{R}^a$. A price vector is $p \in \mathbb{P} \subseteq \mathbb{R}^p$. The exogenous shock follows a Markov-chain with transition function Q mapping from Z to Z. The aggregation function maps $\mathcal{A} : \mathcal{M}(S, \mathcal{S}) \to \mathbb{R}^a$, where $\mathcal{M}(S, \mathcal{S})$ is the space of probability measures on $(S, \mathcal{S})$. The market clearance function maps $\lambda : \mathbb{R}^a \times \mathbb{R}^p \to \mathbb{R}$.

Definition 1. *A Competitive Equilibrium is an agents value function $V_{p,\theta}$; agents policy function g; vector of prices p; measure of agents $\mu_{p,\theta}$; such that*

1. *Given prices p, the agents value function $V_{p,\theta}$ and policy function $g_{p,\theta}$ solve the agents problem*

[4] The parameter vector consists of all the parameters of the model, not just those relating to the agents problem.

$$V_{p,\theta}(x,z) = \max_{y=(y_1,y_2)\in Y_\theta} \left\{ F_{p,\theta}(x,y,z) + \beta \int V_{p,\theta}(y_1',z')Q_\theta(z,dz') \right\} \quad (1)$$

2. Aggregates are determined by individual actions: $A_{p,\theta} = \mathcal{A}(g_{p,\theta}, \mu_{p,\theta})$.

3. Markets clear (in terms of prices): $\lambda(A_{p,\theta}, p, \theta) = 0$.

4. The measure of agents is invariant:

$$\mu_{p,\theta}(x,z) = \int \int \left[\int 1_{x=g^{y_1}_{p,\theta}(\hat{x},z)} \mu_{p,\theta}(\hat{x},z) Q_\theta(z,dz') \right] d\hat{x}dz \quad (2)$$

The dependence of the equilibrium on price p and parameters θ, rather than solely on parameters θ, reflects that analytically we know competitive equilibria exist, but cannot prove uniqueness. The price p is thus used to index the competitive equilibria for a given parameter vector θ. Competitive equilibria, at least in the model of Aiyagari (1994) and it's extension to endogenous labour supply, are however 'almost certaintly' unique (Kirkby, 2016a). We use (p, θ) to index competitive equilibria of the model.

Models fitting this definition include Huggett (1993) and Aiyagari (1994), as well as numerous extensions endogenizing labour supply, introducing taxation, and modeling dynasties. The aggregates in point two generally correspond to the household variables (such as aggregate capital) but in some models may also be aggregates of functions thereof (such as tax revenue). The third point, that prices clear markets, involves rewriting market clearance equations in terms of prices, rather than quantities. A summary of the literature on BHA models, as well as a few other heterogeneous agent models, can be found in Heathcote et al. (2009).

So for example in Aiyagari (1994), a model with which many readers will be familiar, the requirement that aggregates are determined by individual actions is that aggregate capital is the sum of individuals capital holdings, $K = \int k'(x,z)d\mu$. While the market clearance condition is that the interest rate is equal to the marginal product of capital, $\lambda(K,r) = r - \alpha K^{1-\alpha} - \delta = 0$, where K depends on how individual's behaivour reacts to the interest rate.

2.2 Transition Path

We consider transition paths based on a finite sequence of parameter changes. While from a computational perspective these are often thought of as finite horizon, mathematically the true transition path is infinite horizon. For convenience we will think of the transition as beginning in period 0, for T_θ periods there are (announced) changes in θ, and from then on θ takes a fixed value θ_{final} and the economy converges towards some final equilibrium $(p_{final}, \theta_{final})$. In practice we are typically interested in the transition path between two stationary equilibria of a Bewley-Huggett-Aiyagari model. However for the theory and algorithms studied here it turns out to be irrelevant whether or not the initial distribution of agents in the economy is associated with a stationary equilibrium of the economy; for the theory it remains very important that the final distribution is a competitive equilibrium.

So we are interested in transition paths from an initial state, described by a distribution of agents, and the final equilibrium relating to the parametrization θ_{final}, with the price p_{final} being needed to allow for the possibility of multiple equilibra (for a given parameter vector).[5]

Rather than being limited to a single one-off change in parameters we allow for any finite sequence of changes, although only the first change ($t = 0$) will be a surprise, the rest will be pre-announced at time $t = 0$.[6]

Denote by $\{\mu_{initial}, \theta_{final}, p_{final}; \theta_0, \theta_1, \theta_2, ..., \theta_{T_\theta - 1}, \theta_{T_\theta}\}$ the transition path problem to which we wish to find the solution.[7] A solution to the transition path problem will take the form of a sequence of prices $\{p_0, p_1, p_2, ...\}$; the conditions this sequence must satisfy to be considered a solution are given by

Definition 2. *(Transition Path for a BHA Economy) The sequence of prices $\{p_0, p_1, p_2, ...\}$ is a transition path for the transition path problem $\{\mu_{initial}, \theta_{final}, p_{final}; \theta_0, \theta_1, \theta_2, ..., \theta_{T_\theta - 1}, \theta_{T_\theta}\}$ if there exist a sequence of agents value functions $\{V_0, V_1 ...\}$, agents policy functions $\{g_0, g_1, ...\}$, a sequence of agent distributions $\{\mu_0, \mu_1, ...\}$, such that*

1. *$(p_{final}, \theta_{final})$ is a stationary competitive equilibria.*
2. *Agents value functions (and policy functions) solve the agents problem for each $t = 0, 1, 2, ...$ Ie.*

$$V_{p_t,\theta_t}(x, z) = \max_{y=(y_1,y_2)\in Y_{p_t,\theta_t}} \left\{ F_{p_t,\theta_t}(x, y, z) + \beta_t \int V_{p_{t+1},\theta_{t+1}}(y_1', z') Q_{\theta_t}(z, dz') \right\} \quad (3)$$

   and policy functions are the corresponding argmax arguements.

3. *Aggregates are determined by individual actions: $A_t = \mathcal{A}(g_t, \mu_{p_t,\theta_t})$, for all $t = 0, 1, 2, ...$*
4. *Markets clear (in terms of prices): $\lambda(A_t, p_t, \theta_t) = 0$, for all $t = 0, 1, 2, ...$*
5. *Agents period 0 agent distribution is equal to the initial distribution, ie. $\mu_0 = \mu_{initial}$.*

[5] ie. The prices p_{final} are needed in the sense that the possibility of multiple general equilibria for a given parametrization means that the parameters alone are not a sufficient statistic to identify the equilibrium relating to θ_{final}.

[6] Allowing for a sequence of surprise/unannounced changes will not be considered here but would involve repeatedly solving, in a nested manner, the transition path analysed here. Thus any algorithm capable of solving the transition path problem we analyse could, albeit perhaps inefficiently, be applied in a nested manner to solve a sequence of unnanounced changes. Alternatively, a series of surprise changes might be better modelled as a series of shocks to the economy, drawn in a standard manner from a probability distribution that reflects possible future reforms, and so it would be a case of just modelling them as a standard exogenous shock and calculating impulse reponse functions to that series of shocks.

[7] Note that $\theta_{final} = \theta_{T_\theta}$.

6. *The measure of agents evolves according to*

$$\mu_{t+1}(x,z) = \int \int \left[\int 1_{x=g_t^{y_1}(\hat{x},z)} \mu_t(\hat{x},z) Q_{\theta_t}(z,dz') \right] d\hat{x} dz \tag{4}$$

for each $t = 0, 1, 2, ...$

Note that in principle, eg. V_{p_0,θ_0}, depends additionally on the entire transition path of parameters and prices, not just those at time zero, but we supress this notation for tractability. This is also true of many of the other variables, many of which also depend on the initial agents distributions $\mu_{initial}$.

2.3 Finite Horizon Transition Paths

When computing the transition path we typically cannot solve the problem with infinite time periods explicitly, thus some approximation must be used. Here we simply assume that after T ($T > T_\theta$) periods the economy will converge sufficiently close to the final equilibrium associated with $(p_{final}, \theta_{final})$. This is implemented by fixing T, and then checking convergence of the state of the economy after T periods to the final equilibrium. This leads us to the definition of a finite horizon ϵ-transition path problem which reflects the mathematical problem which we will actually solve. We call this an ϵ-transition path as while the theory of BHA models suggests that the true transition problem is infinite horizon it is widely assumed that these ϵ-transition paths are a good finite horizon approximation of the true transition path, in the sense that if they converge as the finite horizon is lengthened then they converge to a true transition path. In what follows for convenience we refer to the ϵ-transition path as simply a finite horizon transition path.

Denote by $\{\mu_{initial}, \theta_{final}, p_{final}; \theta_0, \theta_1, \theta_2, ..., \theta_{T_\theta}\}$ the finite horizon transition path problem to which we wish to find the solution. A solution to the transition path problem will take the form of a sequence of prices $\{p_0, p_1, p_2, ..., p_T\}$; the conditions this sequence must satisfy to be considered a solution are given by

Definition 3. *(Computatable Transition Path for a BHA Economy) The sequence of prices* $\{p_0,p_1,p_2,...,p_T\}$ *is a transition path for the transition path problem* $\{\mu_{initial}, p_{final}, \theta_{final}; \theta_0, \theta_1, \theta_2, ..., \theta_{T_\theta}\}$ *if there exist a sequence of agents value functions* $\{V_0, V_1 ..., V_T\}$*, agents policy functions* $\{g_0, g_1, ..., g_T\}$*, a sequence of agent distributions* $\{\mu_0, \mu_1, ..., \mu_T\}$*, such that*

1. $(p_{final}, \theta_{final})$ *is a stationary competitive equilibria.*

2. *Agents period* T *value function,* V_T*, is equal to the final value function,* $V_T = V_{final}$*.*

3. *Agents value functions (and policy functions) solve the agents problem for each* $t = 0, 1, ..., T-1$*. Ie.*

$$V_{p_t,\theta_t}(x,z) = \max_{y=(y_1,y_2)\in Y_{\theta_{t+1}}} \left\{ F_{p_{t+1},\theta_{t+1}}(x,y,z) + \beta_{t+1} \int V_{p_{t+1},\theta_{t+1}}(y_1', z') Q_{\theta_{t+1}}(z,dz') \right\} \tag{5}$$

and policy functions are the corresponding argmax arguements.

4. *Aggregates are determined by individual actions:* $A_t = \mathcal{A}(g_t, \mu_{p_t,\theta_t})$, *for all* $t = 0, 1, 2, ...T$.

5. *Markets* ϵ*-clear (in terms of prices):* $|\lambda(A_t, p_t, \theta_t)| < \epsilon$, *for all* $t = 0, 1, 2, ...T$.

6. *Agents period* 0 *agent distribution is equal to the initial distribution, ie.* $\mu_0 = \mu_{initial}$.

7. *The measure of agents evolves according to*

$$\mu_{t+1}(x, z) = \int \int \left[\int 1_{x=g_t^{y_1}(\hat{x},z)} \mu_t(\hat{x}, z) Q_{\theta_t}(z, dz') \right] d\hat{x} dz \tag{6}$$

for each $t = 0, 1, 2, ...T - 1$.

8. *Price after* T *periods is sufficiently close to the final price,* $|p_T - p_{final}| \leq \epsilon_{p,final}$.

Notice that because theory suggests that a true finite horizon transition path satisfying the requirements of general equilibrium is unlikely to exist we focus on a finite horizon ϵ-transition path by defining market clearance as satisfied up to some tolerance level ϵ. That is, we require markets ϵ-clear, $|\lambda(A_t, p_t, \theta_t)| < \epsilon$, rather than actual market clearance $\lambda(A_t, p_t, \theta_t) = 0$, for all $t = 0, 1, 2, ...T$. Note that in principle, eg. V_{p_0,θ_0}, depends additionally on the entire transition path of parameters and prices, not just those at time zero, but we supress this notation for tractability. The mathematical problem to be solved when computing the transition path for transition path might thus be described as a two-boundary markov decision problem together with a fixed point problem (the general equilibrium).

We note that this representation of the problem is not unique; for example we might equally think of the solution to the transition path problem in terms of a sequence of agent distributions rather than a sequence of prices.[8] Notice also that while θ_T is equal by definition to θ_{final}, there is no reason why θ_0 and μ_0 need have any particular relationship.

3 Algorithms for Computation

While a number of papers have computed transition paths for BHA models there has so far been little to no systematic study of the performance of various algorithms. We now study the performance of various algorithms. We begin with a description of these algorithms.

All of the algorithms for solving transition path problems studied here take the general form of shooting-algorithms. The solution to a transition path problem takes the form of a sequence of prices (henceforth, price path). The transition path problem to be solved can be thought of as a

[8] The mapping between parameter-price pairs and parameter-agent-distribution pairs is isomorphic for the stationary equilibria of BHA models and an analagous result could likely be proven for transition paths.

final value function, an initial agents distribution, and a sequence of parameter values[9] Shooting-algorithms work by starting from an initial guess for the price path, this is used to generate a new price path (based on partly solving the transtion path problem, as described below). The current (initial) price path and the new price path are then combined to generate an updated price path which is then used as the current price path in the next iteration. This is done until the current price path and the new price path converge. The algorithms studied differ in how they 'combine' the current price path and the new price path to update the current price path to be used in the next iteration.

The following psuedocode presents a general shooting algorithm for solving for the transition path. It assumes that the initial and final stationary equilibria have already been computed and can just be considered as inputs.[10]

Inputs: $p_{initial}, \mu_{initial}, p_{final}, V_{final}, \{\theta_0, ..., , \theta_{T-1}, \theta_T\}$. (Remark: $\theta_T = \theta_{final}$.)
Create initial guess for price sequence $p^0 \equiv \{p_0^0, p_1^0, ..., p_{T-1}^0, p_{final}\}$.
▷ Implemented by setting $p_0 = p_{initial}$, $p_{T/2} = p_{final}$, $p_T = p_{final}$, then use linear interpolation from p_0 to $p_{T/2}$, and from $p_{T/2}$ to p_T.
while $||p^i - p^{i-1}|| >$ 'tolerance constant' **do**
Increment i.
▷ We first iterate backwards on the value functions (based on conditions 2 & 3 of transition path definition).
Let $V_T = V_{final}$.
for $t = T-1, ..., 0$ **do**
Solve $V_t(x, y) = \max_{y=(y_1,y_2)\in Y_{\theta_t}} \left\{ F_{p_t^{i-1},\theta_t}(x, y, z) + \beta_t \int V_{t+1}(y_1', z') Q_{\theta_t}(z, dz') \right\}$
Do this for all (x, z), and also find the policy function $g_t(x, z)$ which is the corresponding argmax.
end for
▷ We now iterate forwards on the agents distribution using the policy functions just computed (based on conditions 6 & 7 of transition path definition).
▷ While doing so also compute the new price sequence.
Let $\mu_0 = \mu_{initial}$
Compute p_0^{new} from μ_0, g_0, p_0^{i-1}, and the aggregation and market clearance conditions.
for $t = 1, ..., T$ **do**
Compute $\mu_t(x, z) = \int\int \left[\int 1_{x=g_{t-1}^{y_1}(\hat{x},z)} \mu_{t-1}(\hat{x}, z) Q_{\theta_{t-1}}(z, dz') \right] d\hat{x}dz$
Do this for all (x, z).

[9] The sequence of parameter values can equally be thought of as indexing a sequence of return functions.

[10] The computation of (the initial and final) stationary competitive equilibria of BHA models is standard (Huggett, 1993; Aiyagari, 1994), the theory underlying the computation well understood (Hopenhayn and Prescott, 1992; Kirkby, 2016b,a), and the performance of various algorithms has been studied for many of the main steps (Aruoba, Fernandez-Villaverde, and Rubio-Ramirez, 2006). We compute them using the default methods of VFI Toolkit (v1.2); namely discretized value function iteration, iterating on the discretized agents distribution, and a simple grid search to find the general equilibrium prices (slow but robust). See vfitoolkit.com.

Compute p_t^{new} from μ_t, g_t, p_t^{i-1}, and the aggregation and market clearance conditions.

end for

$\triangleright$ Combine p^{i-1} and p^{new} to generate p^i. This is where the algorithms will differ.

$p^i = W(p^{i-1}, p^{new})$.

end while

The algorithms studied here differ in terms of $W(p^{i-1}, p^{new})$; how the current price path and the new price path are combined to update the current price path. We consider four different choices of $W(p^{i-1}, p^{new})$ which we will refer to as Constant Weight, Exponentially Decreasing Weights, Opening Window, and Mixture,

- Constant Weight: A simple weighted sum of the current and new prices: $p^i = \Phi_1 * p^{i-1} + (1 - \Phi_1) * p^{new}$

- Exponentially Decreasing Weights: A exponentially decreasing weighting on new path from Φ_1 in first period down to $\Phi_2 * \Phi_1$ in T-1 period.
  $p_t^i = \Phi_1(1 - exp(\frac{tlog(\Phi_2)}{T-1}))p_t^{i-1} + (1 - \Phi_1)(1 - exp(\frac{tlog(\Phi_2)}{T-1}))p_t^{new}$

- Opening Window: A simple weighted sum of the current and new prices, but initially only updating the first few observations. Later iterations move gradually towards updating all prices.
  if $t < \frac{T-1}{\Phi_3 * i}$
  $p_t^i = \Phi_1 * p_t^{i-1} + (1 - \Phi_1) * p_t^{new}$
  else
  $p_t^i = p_t^{i-1}$
  end

- Mixture: A exponentially decreasing weighting on new path from Φ_1 in first period down to $\Phi_2 * \Phi_1$ in T-1 period, but initially only updating the first few observations. Later iterations move gradually towards updating all prices.
  if $t < \frac{T-1}{\Phi_3 * i}$
  $p_t^i = \Phi_1(1 - exp(\frac{tlog(\Phi_2)}{T-1}))p_t^{i-1} + (1 - \Phi_1)(1 - exp(\frac{tlog(\Phi_2)}{T-1}))p_t^{new}$
  else
  $p_t^i = p_t^{i-1}$
  end

The first of these four is currently standard in the literature for computing transtion paths for BHA models and for heterogeneous agent OLG models (Conesa and Krueger, 1999). We arbitrarily use $\Phi_2 = 0.2$ and $\Phi_3 = 3$.

A few observations on the algorithm: of the inputs, $p_{initial}$ and p_{final} are used to create the initial guess. $\mu_{initial}$ and V_{final} play important roles in updating the price sequence at each iteration.[11] p_{final} is deliberately not updated during the iteration step: This is so that if the convergence criteria is spuriously met, in the sense that the price path converges to a price sequence that does not have the correct tail/endpoint, then it will be obvious from looking at the outputed solution due to a jump/kink between p_{T-1} and p_{final}.

The finite-period backwards iteration to solve for the value function and policy function is implemented by pure discretization of the state space. Many alternative numerical methods exist for doing this and the choice is made as it: provides a good benchmark, can be easily implemented as a generalised solution method, and can be easily parallelized on the GPU. The finite-period forwards iteration to solve for the agents distribution is done by discretizing the agents grid. Computing the new prices involves some function evaluations; taking weighted sums over the discretized agents distribution.

One obvious way to get improved performance would be a better initial guess. This approach is not investigated here. Approaches such as using a simpler numerical method to solve the transition path problem (say, eg., linear quadratic value function iteration methods; Díaz-Gímenez (2001)) to generate a better initial guess for the price path sound appealing, but the results of Atolia, Chatterjee, and Turnovsky (2010) suggest they may be misleading.[12] These better initial guesses could then be combined with the algorithms we look at here. In our comparison we choose to stick to a naive linear interpolation between initial and final prices, although the function as implemented in the VFI Toolkit will accept any initial guess for the price path and the interested reader can try out any alternatives they might like.

4 Results from Application of Algorithms

Since the model of Aiyagari (1994) is well known and we are simply solving the obvious extension to include a capital tax we leave its precise description to Appendix A, which also describes the values of the model parameters.

To compare the performance of the four different algorithms/weighting-schemes we compute transition paths for seven different reforms, and for four different weights.

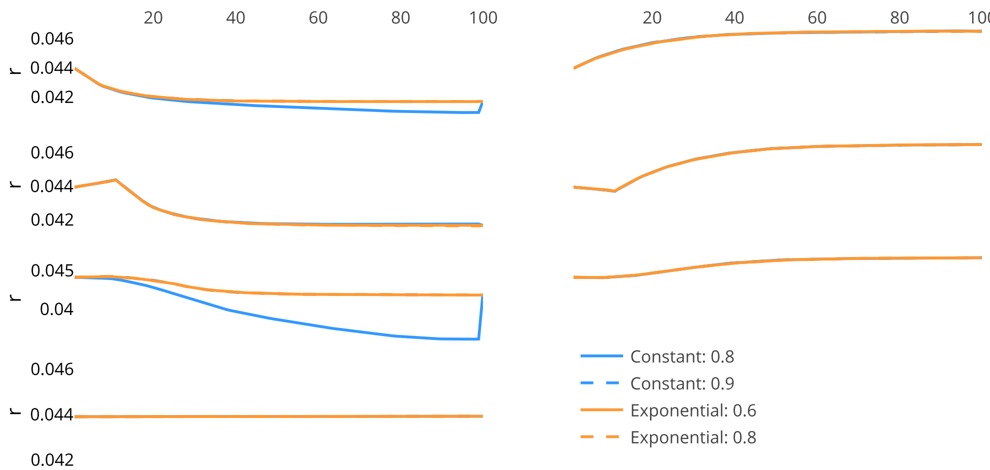
We evaluate seven reforms:

1. A one-off unannounced reduction in capital tax rates.

[11] In our application we use $p_{initial}$ and $\mu_{initial}$ given by the initial recursive competitive equilibrium. In principle however there is no need for $p_{initial}$ and $\mu_{initial}$ to relate to an equilibrium, nor to each other.

[12] Another alternative for better initial guesses may involve some kind of approximation of rational expectations forecasts. This approach has been found to works in models with aggregate shocks (Krusell and Smith, 1998) and for transition paths in OLG models with idiosyncratic but no aggregate shocks (Evans and Phillips, 2014).

Figure 1: Computed Transition Paths on Interest Rates



Notes: Transition Paths for interest rates. Layout of figures for the seven reforms corresponds exactly to those in Table 1. Legends describes which algorithm weights are shown. Periods T=100.

2. A one-off pre-announced reduction in capital tax rates.

3. A series of pre-announced staggered cuts in capital tax rates.

4. 'no-change'.

5. A one-off unannounced increase in capital tax rates.

6. A one-off pre-announced increase in capital tax rates.

7. A series of pre-announced staggered increases in capital tax rates.

We consider four different values of Φ_1, namely 0.6, 0.8, 0.9, 0.95.[13]

The results in terms of runtimes can be seen in Table 1. The use of a constant weight using $\Phi_1 = 0.9$ performs well across a wide variety of situations in terms of speed and robustness. The exponentially-decreasing weights are sometimes competitive with the use of constant weights. While the constant weights can sometimes be made even faster, especially using $\Phi_1 = 0.8$, this occasional improvement in speed comes as the costs of often failing to converge which results in very large runtimes, as seen in Figure 1. The Opening-Window and Mixture are clearly inferior.

The 'No Change' reform was at first surprisingly difficult, with the transition path algorithms failing to converge which turned out to be due to the use of Howards Improvement Algorithm when solving the value function iteration problem for V_{final}. Initially the final value function

[13] We had originially included 0.98 but it gave clearly inferior performace.

Table 1: Run Times for Computing Transition Paths

	Unanticipated Tax Increase				Unanticipated Tax Decrease			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	7701	7680	520	1030	8709	233	464	919
Exponential Weight	339	753	1506	2992	598	1188	2329	4700
Opening Window	7730	7756	1377	1789	8764	1312	1608	2130
Mixture	1506	2049	3015	4838	1772	2534	3879	6499
	Pre-Announced Tax Increase				Pre-Announced Tax Decrease			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	7777	631	559	1067	8897	267	535	1065
Exponential Weight	364	758	1509	2998	593	1183	2364	4755
Opening Window	7782	1143	1381	1768	1153	1326	1622	2161
Mixture	1518	2038	2994	4849	1804	2580	3932	6555
	Series of Tax Increases				Series of Tax Decreases			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	7876	7819	339	629	8819	179	358	745
Exponential Weight	470	756	1515	2998	601	1190	2349	4659
Opening Window	7801	7786	1391	1830	8775	1322	1618	2144
Mixture	1543	2075	3084	4914	1819	2534	3877	6397
			No Change in Taxes					
Scheme/Weight			0.6	0.8	0.9	0.95		
Constant Weight			33	33	33	32		
Exponential Weight			32	33	32	32		
Opening Window			32	32	32	34		
Mixture			32	33	33	32		

Using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$. Time horizon $T = 100$. Run times measures in seconds.

was calculated using Howards improvement algorithm in combination with standard discretized value function iteration (henceforth dVFI). This lead to problems when solving for the transition path for 'no-change'. The use of Howards improvement algorithm does not affect the asymptotic convergence of dVFI. It does however mean that, using the same convergence criterion for stopping iterations, the numerical solution from dVFI and the numerical solution from dVFI with Howards improvement algorithm are subtly different.[14] Solving for the 'no-change' transition path with the final value function calculated using dVFI worked just fine. But solving for the 'no-change' transition path with the final value function calculated using dVFI with Howards improvement algorithm meant that the value function at the beginning of the transition path would shift away from the final value function (and towards the solution to the final value function problem that would have resulted from using just dVFI). As a result the 'no-change' transition path with the final value function calculated using dVFI with Howards improvement algorithm would shift away from the initial steady-state at the beginning of the transition path; which we know analytically

[14] Using a convergence criterion that the absolute difference between consecutive iterations of the value function was less than 10^{-9} dVFI with and without Howards improvement algorithm lead to solutions that differred by something between 1 and 10 times that amount.

Table 2: Further Run Times for Computing Transition Paths

	Unanticipated Tax Increase		Unanticipated Tax Decrease	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	7646	520	204	410
Exponential Weight	7703	335	409	517
Exponential Post Reforms	335	340	519	520
	Pre-Announced Tax Increase		Pre-Announced Tax Decrease	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	624	552	234	469
Exponential Weight	657	368	443	528
Exponential Post Reforms	395	396	550	549
	Series of Tax Increases		Series of Tax Decreases	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	7892	345	177	364
Exponential Weight	7922	7955	511	635
Exponential Post Reforms	7979	8187	647	652
		No Change in Taxes		
Scheme/Weight		0.8	0.9	
Constant Weight		30	31	
Exponential Weight		30	30	
Exponential Post Reforms		30	30	

Using grid sizes $n_k = 512$, $n_z = 21$, $n_p = 251$. Time horizon $T = 100$. Run times measures in seconds. Ten repetitions were run, since run times across all repetitions were within 5 seconds of each other only the first is reported. Run times over 7000 seconds were all cases where algorithm terminated due to reaching max number of iterations (not convergence criterion).

is incorrect (violates the definition of the steady-state). In practice this was easily resolved by simply continuing to take advantage of the speed-up improvements for dVFI that come from using Howards improvement algorithm, and then turning off Howards improvement algorithm and solely using dVFI once the value function iterations got within an order of magnitude of the convergence criterion. We conclude from this that robustness of any algorithm for transition paths is going to rely on it's using the same method for calculating the value function (and agents distribution) along the transition as was used for the initial and final steady-states.

Appendix B extends the model to include an endogenous labour supply decision. Our conclusions based on the results for the transition path algorithms in terms of their robustness and speed remain unchanged.

5 Conclusion

We considered a variety of algorithms for solving general equilibrium transition paths in Bewley-Huggett-Aiyagari models. A comparison of various algorithms/weightings to solve for the transition paths of a variety of capital tax reforms finds that the current literature standard of constant weights performs well in terms of runtime and robustness; specifically when using a weight of 0.9 on the old price path.

References

Omer Tugrul Acikgov. Transitional dynamics and long-run optimal taxation under incomplete markets. Working Paper, 2015.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. Quarterly Journal of Economics, 109(3):659–684, 1994.

S. Rao Aiyagari. Optimal capital income taxation with incomplete markets, borrowing constraints, and constant discounting. Journal of Political Economy, 103(6):1158–1175, 1995.

S. Aruoba, Jesus Fernandez-Villaverde, and Juan Rubio-Ramirez. Comparing solution methods for dynamic equilibrium economies. Journal of Economic Dynamics and Control, 30(12):2477–2508, 2006.

Manoj Atolia, Santanu Chatterjee, and Stephen J. Turnovsky. How misleading is linearization? evaluating the dynamics of the neoclassical growth model. Journal of Economic Dynamics and Control, 34(9):1550–1571, 2010.

Alan Auerbach and Joel Slemrod. The economic effects of the tax reform act of 1986. Journal of Economic Literature, 35(2):589–632, 1997.

Ozan Bakis, Baris Kaymak, and Markus Poschke. Transitional dynamics and the optimal progressivity of income redistribution. Review of Economic Dynamics, 18(3):679–693, 2015.

Christophe Chamley. Optimal taxation of capital income in general equilibrium with infinite lives. Econometrica, 54(3):607–622, 1986.

Juan Carlos Conesa and Dirk Krueger. Social security reform with heterogeneous agents. Review of Economic Dynamics, 2(4):757–795, 1999.

Juan Carlos Conesa, Sagiri Kitao, and Dirk Krueger. Taxing capital? not a bad idea after all! American Economic Review, 99(1):25–48, 2009.

Wouter Den Haan, Michel Julliard, and Kenneth Judd. Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. Journal of Economic Dynamics and Control, 34(1):1–3, 2010.

J. Díaz-Gímenez. Linear quadratic approximations: An introduction. In R. Márimon and A. Scott, editors, Computational Methods for the Study of Dynamic Economies, chapter 2. Oxford University Press, 2001.

David Domeij and Jonathan Heathcote. On the distributional effects of decreasing capital taxes. International Economic Review, 45(2):523–544, 2004.

Richard Evans and Kirk Phillips. Olg life cycle model transition paths: Alternate model forecast method. Computational Economics, 43(1):105–131, 2014.

Hans Fehr and Fabian Kindermann. Taxing capital along the transition not a bad idea after all? Journal of Economic Dynamics and Control, 51:64–77, 2015.

Martin Floden. Why are capital income taxes so high? Macroeconomic Dynamics, 13(3):279–304, 2009.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni Violante. Quantitative macroeconomics with heterogeneous households. Annual Review of Economics, 1(5):319–354, 2009.

Hugo Hopenhayn and Edward C. Prescott. Stochastic monotonicity and stationary distributions for dynamic economies. Econometrica, 60(6):1387–1406, 1992.

Mark Huggett. The risk-free rate in heterogenous-agent incomplete-insurance economies. Journal of Economic Dynamics and Control, 17:953–969, 1993.

Kenneth Judd. Redistributive taxation in a simple perfect foresight model. Journal of Public Economics, 28:59–83, 1985.

Tim Kehoe and David Levine. Comparative statics and perfect foresight in infinite horizon economies. Econometrica, 53(2):433–453, 1985.

Robert Kirkby. Bewley-huggett-aiyagari models: Computation and simulation. (Submitted), 2016a.

Robert Kirkby. Convergence of discretized value function iteration. Computational Economics, 2016b.

Sagiri Kitao. Entrepreneurship, taxation and capital investment. Review of Economic Dynamics, 11(1):44–69, 2008.

Per Krusell and Anthony Smith. Income and wealth heterogeneity in the macroeconomy. Journal of Political Economy, 106(5):867–896, 1998.

Robert E. Lucas. Supply-side economics: An analytical review. Oxford Economic Papers, 42(2): 292–316, 1990.

Josep Pijoan-Mas. Precautionary savings or working longer hours? Review of Economic Dynamics, 9(2):326–352, 2006.

Nancy Stokey, Robert E. Lucas, and Edward C. Prescott. Recursive Methods in Economic Dynamics. Harvard University Press, 1989.

A The Specific Model

In the model of Aiyagari (1994) infinitely lived households face a stochastic income — due to exogenous stochastic labour supply — and make consumption-savings decisions; given an interest rate. So the exogenous shock (z) is the labour supply h, the endogenous state (x) is capital holdings k, and the decision variable (y) is next periods capital k'. The state of a household is their current capital holding and their exogenous labour supply shock, (k, h). Individual household capital holdings, given an interest rate, aggregate to give aggregate capital holdings. The market clearance condition is that the interest rate will be determined by perfect competition in the goods market together with a representative firm with Cobb-Douglas production function. A general equilibrium is that an interest rate, which determines household capital holdings, which in turn by the market clearance condition determine an interest rate, and that this later interest rate is the same as the first. We add a capital income tax (τ_k), modelled as a tax on the interest rate on capital. For comparability with Aiyagari (1995) a labour tax is also added. To simplify the transition paths to be a single dimension we assume the government simply burns all the tax revenue; otherwise we would have to add some kind of assumption about how the government deals with fiscal imbalances between revenue and consumption over the transition path.[15] In short,

Definition 4. *A Competitive Equilibrium is an agents value function $V(k,h)$; agents policy function $k' = g(k,h)$; an interest rate r and wage w; aggregate capital K and labour H; and a measure of agents $\mu(k,h)$; such that*

1. *Given prices r & w, the agents value function $V(k,h)$ and policy function $k' = g(k,h)$ solve the agents problem*

$$\begin{aligned} V(k,h) = &\max_{k'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \int V(k',h')Q(h,dh') \right\} \\ &s.t. \quad c + k' = wh + (1 + (1-\tau_k)r)k \\ &\quad ln(h') = \rho ln(h) + \epsilon_h, \epsilon_h \sim N(0, \sigma^2_{\epsilon_h}) \\ &\quad c \geq 0,\ k' \geq \underline{k} \end{aligned}$$

2. *Aggregates are determined by individual actions: $K = \int k d\mu(k,h)$, and $H = \int h d\mu(k,h)$*

3. *Markets clear (in terms of prices): $r - (\alpha K^{\alpha-1}H^{1-\alpha} - \delta) = 0$.*

4. *The measure of agents is invariant:*

$$\mu(k,h) = \int \int \left[\int 1_{k=g(\hat{k},h)} \mu(\hat{k},h)Q(h,dh') \right] d\hat{k}dh \tag{7}$$

[15] Options would include that the government runs a balanced budget every period, that the government accumulates debt over the transition, or some combination of the two.

where h is the labour supply shock which takes values in $Z = \{h_1, ..., h_{n_h}\}$ and evolves according to Markov transition function $Q(h, h')$ (which is here given by $ln(h') = \rho ln(h) + \epsilon_h$). Note that the wage is residualy determined by r.[16] The market clearance condition is more commonly expressed as $r = \alpha K^{\alpha-1} H^{1-\alpha} - \delta$, that the interest equals the marginal product of capital (minus the depreciation rate). Since $H = E(h) = 1$ the Cobb-Douglas production function is really only based on aggregate capital (in the sense that H is a fixed constant).

The optimal capital tax rate in this model is positive (Aiyagari, 1995).[17]

Following (the working paper version of) Aiyagari (1995) and Aiyagari (1994) we calibrate the model as $\gamma = 3$, $\rho = 0.9$, $\sigma_{\epsilon_h} = 0.2$, $\delta = 0.08$, and $\alpha = 0.36$.

[16] The wage, which is given by the derivative of the Cobb-Douglas production with respect to labour, can be rewritten as a function of the interest rate and the parameters of the production function.

[17] A positive capital tax reduces aggregate capital towards the efficient level. Without a capital tax the aggregate capital stock is inefficiently high due to precautionary savings arising from incomplete markets. Aiyagari (1995) shows that the inefficiently high aggregate capital stock level cannot be reduced to the efficient level simply by creating a substitute risk-free asset in the form of government debt, which if possible would obviate the need for non-zero capital taxation to achieve the efficient level of aggregate capital; in contrast to many OLG models where government debt can be used in this manner.

B Extension of Application to Endogenous Labour supply

We extend the model of Aiyagari (1994) to include endogenous labour supply, following the set up of Pijoan-Mas (2006). We then look at computing the transition paths for the same reforms using the same algorithms as were considered in for the model of Aiyagari (1994). Tables 3 and 4 report the analagous run time results to those in Tables 1 and 2. Figure 2 reports the analagous results showing the interest rate paths to those in Figure 1. The findings for the extension to endogenous labour supply closely mirror those without and the conclusions drawn in terms of which algorithms work best are exactly the same.

We begin with a description of the extension of the model to include endogenous labour supply. Infinitely lived households face a stochastic earnings opportunities — an exogenous shock process on efficiency labour units — and make consumption-savings and consumption-leisure decisions; given an interest rate and wage rate. So the exogenous state (z) is efficiency labour units, the endogenous state (x) is capital holdings k, and the decision variables (y) are next periods capital k' and hours worked h. The state of a household is their current capital holding and their exogenous labour supply shock, (k, z). Individual household capital holdings, given an interest rate, aggregate to give aggregate capital holdings. Individual hours worked and efficiency labour units together aggregate to give aggregate labour supply (measured in efficiency labour units). The market clearance condition is that the interest rate and wage (per efficient labour unit) will be determined by perfect competition in the goods market together with a representative firm with Cobb-Douglas production function. A general equilibrium is an interest rate (and wage), which determines household capital holdings and labour supply, which in turn by the market clearance condition determine an interest rate (and wage), and that this later interest rate (and wage) is the same as the first. We add a capital income tax (τ_k), modelled as a tax on the interest rate on capital, and a labour tax is also added. To simplify the transition paths to be a single dimension we assume the government simply burns all the tax revenue; otherwise we would have to add some kind of assumption about how the government deals with fiscal imbalances between revenue and consumption over the transition path.[18] In short,

Definition 5. *A Competitive Equilibrium is an agents value function $V(k, z)$; agents policy function $(h, k') = g(k, z)$; an interest rate r and wage w; aggregate capital K and labour H; and a measure of agents $\mu(k, z)$; such that*

1. *Given prices r & w, the agents value function $V(k, z)$ and policy function $(h, k') = g(k, z)$*

[18] Options would include that the government runs a balanced budget every period, that the government accumulates debt over the transition, or some combination of the two.

solve the agents problem

$$V(k,z) = \max_{h,k'} \left\{ \frac{c^{1-\gamma_c}-1}{1-\gamma_c} + \chi \frac{(\ell-h)^{1-\gamma_h}-1}{/} 1-\gamma_h + \beta \int V(k',z')Q(z,dz') \right\}$$
$$s.t. \quad c + k' = whz + (1+(1-\tau_k)r)k$$
$$ln(z') = \rho ln(z) + \epsilon_z, \epsilon_z \sim N(0, \sigma^2_{\epsilon_z})$$
$$c \geq 0,\ k' \geq \underline{k},\ h \in [0,1]$$

2. *Aggregates are determined by individual actions:* $K = \int k d\mu(k,z)$, *and* $L = \int hz d\mu(k,z)$

3. *Markets clear (in terms of prices):* $r - (\alpha K^{\alpha-1} L^{1-\alpha} - \delta) = 0$, $w - ((1-\alpha)K^{\alpha} L^{-\alpha}) = 0$

4. *The measure of agents is invariant:*

$$\mu(k,z) = \int \int \left[\int 1_{k=g(\hat{k},z)} \mu(\hat{k},z) Q(z,dz') \right] d\hat{k} dz \quad (8)$$

where z is the labour efficiency units shock which takes values in $Z = \{h_1, ..., h_{n_z}\}$ and evolves according to Markov transition function $Q(z,z')$ (which is here given by $ln(z') = \rho ln(z) + \epsilon_z$). Note that the wage is residualy determined by r.[19] The market clearance conditions are more commonly expressed as $r = \alpha K^{\alpha-1} H^{1-\alpha} - \delta$, that the interest equals the marginal product of capital (minus the depreciation rate), and $w = (1-\alpha)K^{\alpha}L^{-\alpha}$, that the wage per efficiency labour unit equals the marginal product of an efficiency labour unit.

The optimal capital tax rate in this model is will be lower in this model than in the model with exogenous labour (as precautionary savings are now less important due to alternative channel of precautionary labour supply).

Following Pijoan-Mas (2006) we calibrate the model as $\beta = 0.945$,$\gamma_c = 1.458$, $\gamma_l = 2.833$, $\ell = 1$, $\chi = 0.856$, $\delta = 0.0834$, $\alpha = 0.36$, $\rho = 0.95$, $\sigma_{\epsilon_z} = 0.21$.

Applying the exact algorithms described in Section 3 to the exact same reforms described in Section 4 we find the same general results as for the model with exogenous labour supply, as were summarized in Section 4. Table 3 provides some basic runtimes. Figure 2 shows the resulting transition paths for the interest rate. Table 4 provides further runtimes based on ten replications each for the algorithms that performed best the first time round.

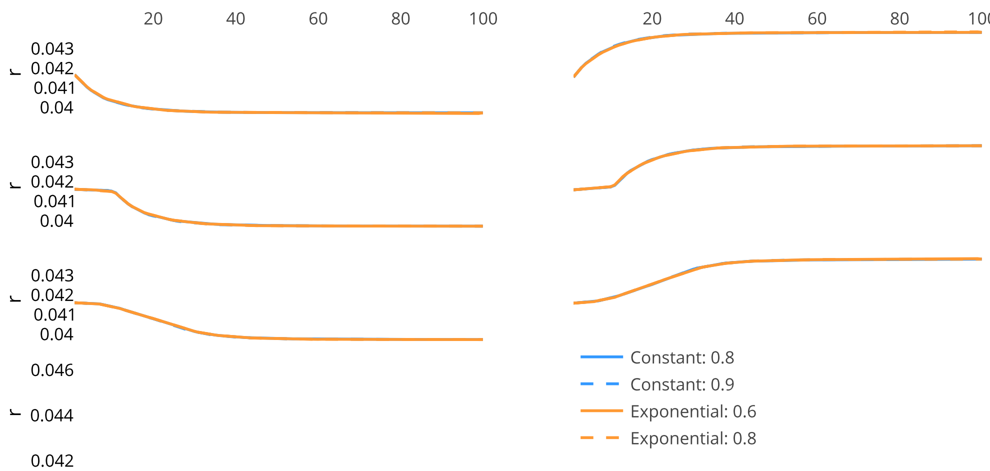
[19] The wage, which is given by the derivative of the Cobb-Douglas production with respect to labour, can be rewritten as a function of the interest rate and the parameters of the production function.

Table 3: Run Times for Computing Transition Paths

	Unanticipated Tax Increase				Unanticipated Tax Decrease			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	2404	390	789	1519	920	435	809	1618
Exponential Weight	980	1913	3873	7707	1136	2325	4599	9256
Opening Window	1925	2118	2556	3353	2057	2378	2870	3798
Mixture	2812	3843	5855	9813	3135	4387	6777	11498
	Pre-Announced Tax Increase				**Pre-Announced Tax Decrease**			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	14836	347	645	1336	921	380	812	1629
Exponential Weight	989	1932	3919	7786	1140	2344	4617	9277
Opening Window	1886	2134	2581	3368	2066	2394	2877	3804
Mixture	2826	3863	5901	9871	3149	4403	6793	11469
	Series of Tax Increases				**Series of Tax Decreases**			
Scheme/Weight	0.6	0.8	0.9	0.95	0.6	0.8	0.9	0.95
Constant Weight	1646	298	642	1340	977	379	813	1631
Exponential Weight	993	1935	3869	7804	1141	2284	4623	9251
Opening Window	1883	2133	2585	3377	2063	2391	2881	3807
Mixture	2839	3879	5909	9882	3154	4402	6801	11472
				No Change in Taxes				
Scheme/Weight			0.6	0.8	0.9	0.95		
Constant Weight			60	60	60	58		
Exponential Weight			58	59	59	59		
Opening Window			58	59	59	59		
Mixture			59	58	59	59		

Using grid sizes $n_l = 51$, $n_k = 351$, $n_z = 21$, $n_p = 251$. Time horizon $T = 100$. Run times measures in seconds.

Figure 2: Computed Transition Paths on Interest Rates



Notes: Transition Paths for interest rates in extension of model to endogenous labour supply. Layout of figures for the seven reforms corresponds exactly to those in Table 3. Legends describes which algorithm weights are shown in the Figures. Periods T=100.

Table 4: Further Run Times for Computing Transition Paths

	Unanticipated Tax Increase		Unanticipated Tax Decrease	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	403	806	441	820
Exponential Weight	794	996	936	1207
Exponential Post Reforms	997	998	1211	1212
	Pre-Announced Tax Increase		Pre-Announced Tax Decrease	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	350	650	386	827
Exponential Weight	803	1003	943	1218
Exponential Post Reforms	1003	1004	1164	1161
	Series of Tax Increases		Series of Tax Decreases	
Scheme/Weight	0.8	0.9	0.8	0.9
Constant Weight	301	652	387	829
Exponential Weight	840	1056	942	1162
Exponential Post Reforms	1053	1056	1224	1220
		No Change in Taxes		
Scheme/Weight		0.8	0.9	
Constant Weight		60	59	
Exponential Weight		59	60	
Exponential Post Reforms		59	60	

Using grid sizes $n_k = 351$, $n_z = 21$, $n_p = 251$. Time horizon $T = 100$. Run times measures in seconds. Ten repetitions were run, since run times across all repetitions were within 5 seconds of each other only the first is reported. Run times over 7000 seconds were all cases where algorithm terminated due to reaching max number of iterations (not convergence criterion).

TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI

VICTORIA

UNIVERSITY OF WELLINGTON