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The invisible polluter: Can regulators save consumer surplus?

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Abstract

Consider an electricity market populated by competitive agents using thermal generating units. Such generation involves the emission of pollutants, on which a regulator might impose constraints. Transmission capacities for sending energy may naturally be restricted by the grid facilities. Both pollution standards and transmission capacities can impose several constraints upon the joint strategy space of the agents. We propose a coupled constraints equilibrium as a solution to the regulator's problem of avoiding both congestion and excessive pollution. Using the coupled constraints' Lagrange multipliers as taxation coefficients the regulator can compel the agents to obey the multiple constraints. However, for this modification of the players' payoffs to induce the required behaviour a coupled constraints equilibrium needs to exist and must also be unique. A three-node market example with a *dc* model of the transmission line constraints described in [8] and [2] possesses these properties. We extend it here to utilise a two-period load duration curve and, in result, obtain a two-period game. The implications of the game solutions obtained for several *weights*, which the regulator can use to vary the level of generators' responsibilities for the constraints' satisfaction, for consumer and producer surpluses will be discussed.

Keywords: Coupled constraints; generalised Nash equilibrium; electricity production; electricity transmission
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1 Introduction

The aim of this paper is to examine the impact of pollution standards on electricity generators already subjected to grid facility restrictions.[1] We consider an electricity

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market populated by competitive agents using thermal generating units. Such generation emits pollutants, on which a regulator might wish to impose constraints. Transmission capacity for sending energy may naturally be restricted by the grid facilities. Both pollution standards and transmission capacity can be defined as constraints upon the joint strategy space of the agents.

We notice that the setup of the problem in this paper is similar to that discussed in [3]. Here, however, we utilise a two-period load duration curve and allow for imposition of the environmental constraints on the two-period joint emissions, while the analysis in [3] was confined to one period. We also make explicit in this paper the relationship between a solution to the problem and the *weights*, which the regulator may use to distribute the responsibility for the joint constraints' satisfaction, among the generators and/or periods, see Definition 3.1.

We follow [19], [13], [2], [10], and also [3], and use a *coupled constraints equilibrium* as a solution concept for the discussed problem. Under this solution concept the regulator can compute (for sufficiently concave games) the generators' outputs that are both unilaterally non-improvable (Nash) and which satisfy the constraints imposed on the joint strategy space.

If the regulator can impose penalties on the generators for violation of the joint constraints then the game becomes "decoupled", once the players incorporate the penalties in their payoff functions. If so, the players implement the *coupled constraints equilibrium* to avoid fines associated with both congestion and excessive pollution. These penalties that prevent excessive generation are computed using the coupled constraints Lagrange multipliers. However, for this modification of the players' payoffs to induce the required behaviour, a coupled constraints equilibrium needs to exist and be unique for a given distribution of the responsibilities for the joint constraints satisfaction, among the generators and periods. A three-node bilateral market example with a *dc* model of the transmission line constraints (described in [2]) possesses these properties and will be used in this paper to discuss and explain the behaviour of agents subjected to coupled constraints.

The analysis conducted in this paper should be particularly useful to regional governments interested in assessing the impact of environmental regulation on electricity generation. The case study considered shows significant market distortion and re-allocation of surplus as a result of the imposition of a pollution constraint. We also show that by altering the degrees of responsibility for the joint constraints satisfaction, among the generators and periods, the regulator may help to "save" the consumer surplus.

For the results we use NIRA, which is a piece of software designed to min-maximise the Nikaido-Isoda function and thus compute a coupled constraints equilibrium (see [14]). We also notice that a coupled constraints equilibrium could be obtained[2] as a solution to a quasi-variational inequality[3] (see [9], [17]) or gradient *pseudo-norm* minimisation (see [19], [5], [6]).

[2]We refer to [11] for a review on numerical solutions to coupled-constraint equilibria.

[3]Some comparisons between NIRA and a method suitable for quasi-variational problems, which consists of a sequence of solutions of linear complementarity problems, are provided in [15]. While both methods deliver the same solution, NIRA performs less efficiently on the chosen (linear) example. As was observed in [22], NIRA is general being suitable for non-smooth games and can be outperformed when games are "smooth".

What follows is a brief outline of what this paper contains. In section 2 a model of a bilateral electricity market game is presented. Section 3 briefly explains the idea of a coupled constraints equilibrium and the algorithm that will be used to find it. Sections 4, 5 and 6 present the parameters of the case study and the results of our analysis. In section 7 an economic interpretation is given to the results. The concluding remarks summarise our findings.

2 The model

2.1 A game with constraints

An electricity market is a system for effecting the purchase and sale of electricity, where the interaction between supply and demand sets the market price. Transactions are typically cleared and settled by the market operator or by a special-purpose independent entity charged exclusively with that function. If the offers by the generators and the demand bids are matched bilaterally then the market is known as bilateral. We focus on this type of electricity market in this paper.

Transmission systems connect generators and consumer loads in an electricity network and they are operated to allow for continuity of supply. Transmission networks can experience bottlenecks; in addition, the authorities usually establish pollution limits on the generators' emissions. These constraints limit the production of the generators and, consequently, their profits. In the following subsections we explain how the generators optimise their production and how the network and environmental constraints affect their profits in a coupled constraints game.

2.2 Generator's problem

We assume no arbitrage (existence of marketers that can buy and sell power from producers and consumers) and a linear *dc* representation of the network.[4]

Each generator is maximising the sum of payoffs over a horizon T divided into periods of length Δ^t. The period length corresponds to the time within which the demand for electricity is considered constant according to the *load duration curve*[5].

Each company $f = 1, \ldots, F$ owns several generating units $g = 1, \ldots, G$ distributed throughout the network composed of nodes $i, j = 1, \ldots, N$. The cost of running unit g that belongs to company f, placed at node i for period Δ^t and whose power in this period is P^t_{fgi} MW, is $C(P^t_{fgi})\Delta^t$. The maximum capacity of generator is $\overline{P}_{fgi}$.

Consumers buy q^t_i MWh of energy at node i in period t. At each node, linear demand functions are assumed to be of the form $p^t_i(\sigma^t_i) = a^t_i - \dfrac{a^t_i}{b^t_i}\,\sigma^t_i$ where σ^t_i is the hourly demand in period t defined as $\sigma^t_i = q^t_i/\Delta^t$ measured in MW (we will later

[4]See [4] for technical details.

[5]The load duration curve is a curve of loads, plotted in a descending order of magnitude, against the time intervals for a specified period. Load duration curves are profiles of system demand that can be drawn for specified periods of time (e.g., daily, monthly, yearly)

refer to σ_i^t as demand for power). Price $p_i^t(\sigma_i^t)$ is expressed in \$/MWh and a_i^t and b_i^t are the price and power intercepts for (inverse) demand law at node i, respectively.

Energy s_{fj}^t MWh is sold by company f to consumers at node j in period t. Market clearing is such that the condition $\sum_f s_{fj}^t = q_j^t$ holds. Also, an energy balance per period is imposed on each company: $\sum_{i,g} P_{fgi}^t \Delta^t = \sum_j s_{fj}^t$. Given that (and remembering that demand is constant within Δ^t), each company f chooses generation P_{fgi}^t and sales s_{fi}^t to maximise profit (\$), which is equal to revenue minus generation costs:

$$\max \sum_{j,t} \left[a_j^t - \frac{a_j^t}{b_j^t} \left(\sigma_{fj}^t + \sum_{k \neq f} \sigma_{kj}^t \right) \right] s_{fj}^t \; - \sum_{i,g,t} C(P_{fgi}^t) \Delta^t \tag{1}$$

subject to:

$$\left. \begin{array}{cl} 0 \leq P_{fgi}^t \leq \overline{P}_{fgi}, & \forall \text{ nodes } i, \text{ generators } g, \text{ periods } t \\ \sum_j s_{fj}^t = \sum_{i,g} P_{fgi}^t \Delta^t, & \text{for each firm } f, \forall \text{ periods } t \\ \sum_f s_{fj}^t = q_j^t, & \text{for each node } j, \forall \text{ periods } t \end{array} \right\} \tag{2}$$

We are interested in a non-cooperative solution to the game at hand. This means that we are looking for a distribution of generation and corresponding payoffs such that no player can improve his own payoff by a unilateral action. Bearing in mind that the solution is required to satisfy constraints, it will need to be understood as a "generalised" Nash-Cournot equilibrium as it is called in [17], or a coupled constraints equilibrium as we call it. We explain this concept in section 3.

Notice that problem (1), (2) is time-decomposable because the generators cannot accumulate power across the periods. However, a generator problem would *not* enjoy this feature if an inter-temporal constraint were imposed.

2.3 Transmission constraints

The generating units and the nodes are connected by transmission lines, forming a network. The lines provide a path to transmit the power produced by the generators to the nodes for consumption. The power flowing through the lines is subject to thermal line limits. These limits are set in both directions of the flow in a line, which is why the absolute value is used in equation (3) below. To represent the topology of the network in this game it is necessary to select a node as the reference node. This is called the "slack" node or "swing" node (see [4] for details). The following equation expresses the flow going through the lines in period t as a linear function of the power injected at the nodes; this is called the *dc* approximation of the power flow:

$$\mathbf{P}_{i\to j}^t \equiv \left[B_d \cdot A^T \cdot B^{-1} \right] \cdot \mathbf{P}^t, \quad \text{with} \quad |\mathbf{P}_{i\to j}^t| \leq \overline{\mathbf{P}}_{i\to j}, [6] \tag{3}$$

where the variables and parameters are as follows:

[6] Note that, throughout this paper, we use the notation $|\mathbf{P}_{i\to j}^t|$ to denote a matrix with each element being the absolute value of the corresponding element of $\mathbf{P}_{i\to j}^t$. It is not intended to represent the norm of a vector.

$\mathbf{P}^t_{i\to j}$ is a column vector whose number of rows is equal to the number of lines of the network. Each element represents the flow through the line $i-j$ in period t, measured in MW.

B_d is the diagonal branch susceptance matrix, whose number of rows and columns is equal to the number of lines in the network. The diagonal terms of B_d are the susceptances of the lines (susceptance is the inverse of reactance). The reactances are expressed in *per-unit* relative to the impedance base[7] value.

A is the node-branch incidence matrix; its dimensions are the number of nodes minus one (slack node) by the number of lines. The values of A are equal to $+1$ if the line $i-j$ starts at node i, and -1 if it ends at node j.[8]

B is the diagonal node-to-node susceptance matrix; its dimensions are equal to the number of nodes minus one (slack node). The diagonal terms b_{ii} are equal to the sum of the susceptances of the lines that are connected to node i, and the terms b_{ij} are equal to the negative of the susceptances of the lines that connect node i and node j.

$\mathbf{P}^t$ is a column vector whose dimension is equal to the number of nodes minus one (the slack node). Its components are of the form $(P^t_{fgi} - q^t_i/\Delta^t)$, representing the power in MW injected (generation minus demand) in each of the nodes in period t, except for the slack node.

$\overline{\mathbf{P}}_{i\to j}$ is a column vector whose number of rows is equal to the number of lines of the network. Each of its elements represents the thermal limit (maximum active power that can flow through a line) of a line in MW.

2.4 Pollution constraints

The thermal generation of electricity releases several contaminants into the atmosphere. The overall goal of reducing the emission of pollutants has to be expressed as a constraint for the overall production of all generating units. There are three main types of emissions: CO_2, SO_2 and NO_x. The general expression of the pollution constraint for emissions of type ℓ is (see [18]):

$$\sum_{h,t} \Delta^t \left[\alpha_{h\ell} + \beta_{h\ell} P_h + \gamma_{h\ell} P_h^2\right] \le \overline{K}_\ell \,, \quad \ell = 1, \dots L \tag{4}$$

where L can denote the number of noxious substances, for which restrictions are to be enforced or it may refer to the number of locations at which substance limits need to be observed. Δ^t is the duration in hours of period t of the load duration curve and P_h is the power output of generating unit h. The unit will be the g-th generator of firm f located at node i. However, for pollution generation and its constraint, the location of h relative to monitor ℓ is relevant. Coefficients $\alpha_{h\ell}$,

[7] The $per-unit$ value of any quantity is defined as the ratio of the quantity to its base value, expressed as a decimal. See [20] for details.

[8] The indices here refer to the nodes rather than the matrix elements.

$\beta_{h\ell}$, and $\gamma_{h\ell}$ correspond to emissions discharged by unit h measured as pollution at location ℓ ("type" ℓ). Amount $\overline{K}_\ell$ is the maximum allowed pollution of type ℓ during all periods, usually measured in lb or Ton.

In (4), we have chosen to restrict the maximum allowed pollution of type ℓ during *all* periods rather than per period, mainly to reflect the popular policy of many governments to delimit the allowable pollution in annual *i.e.*, *all*-period terms. This introduces a *dynamic* aspect to the game: given a binding value of $\overline{K}_\ell$ the firms will consider tradeoffs between generating power in one period against generating it in another period so that the constraint is satisfied and their payoffs are maximised.

Each term of the left hand side of expression (4) can be interpreted as a steady-state solution to the partial differential equation describing the dispersion of pollution from a point source (see [1] for the *integrated Gaussian puff diffusion model*). In this paper, the coefficients of (4) have been calibrated following [18]).

The above constraint reflects the regulator's concern for substance ℓ concentration at a selected (representative) location. More constraints of type (4) could be added to the problem formulation for more locations, at which the regulator would want to enforce compliance. If needed, the constraints can be defined as limits *per periods* and/or *per generator*, should such limits be known and adequate monitoring facilities exist.

3 Constrained equilibria

3.1 Coupled constraints equilibria

A coupled constraints equilibrium (CCE) is an extension of a standard Nash equilibrium in which players' strategy sets are allowed to depend upon other players' strategies. Coupled constraints equilibria are also known as generalised Nash equilibria. The competition between electricity generating firms subject to constraints described above in section 2 is an example of such a problem. Analytical solutions to CCE problems are not normally possible so section 3.2 describes a numerical method for solving some such problems.

Coupled constraints equilibria are particularly useful in a class of problems where competing agents are subjected to regulation. Many electricity market and environmental problems belong to this class. CCE allows modelling of a situation in which the actions of one player condition how 'big' the actions of other players can be. Constraints in which the actions of one player do not affect the action space of another (as in Nash equilibrium problems) are called uncoupled.

In our problem there are two such sets of coupled constraints: the line constraints and the environmental constraints. In both cases a limit is placed on a measurable variable — the flow of electricity through a particular line or the ambient pollution levels — and the actions of the players are constrained to jointly satisfy these limits.

In these games the constraints are assumed to be such that the resulting collective action set X is a closed convex subset of $\mathbb{R}_m$. If X_f is player-f's action set, $X \subseteq X_1 \times \cdots \times X_F$ is the collective action set (where $X = X_1 \times \cdots \times X_F$ represents the special case in which the constraints are uncoupled).

Consider the solution to this game represented by the collective action $\mathbf{x}^*$ where players' payoff functions, ϕ_f, are continuous in all players' actions and concave in their own action[9]. The Nash equilibrium can be written as

$$\phi_f(\mathbf{x}^*) = \max_{\mathbf{x} \in X} \phi_f(y_f|\mathbf{x}^*) \tag{5}$$

where $y_f|\mathbf{x}^*$ denotes a collection of actions where the fth agent "tries" y_f while the remaining agents continue to play the collective action $\mathbf{x}^*$. Note that $\mathbf{x}^*$ is a column vector with elements x_g, $g = 1, 2, \ldots, f-1, f+1, \ldots, F$. At $\mathbf{x}^*$ no player can improve his own payoff through a unilateral change in his strategy so $\mathbf{x}^*$ is a Nash equilibrium point. If X is a closed and strictly convex set defined through coupled constraints (like (4)) then $\mathbf{x}^*$ is a CCE.

3.2 NIRA

Games with coupled constraints rarely allow for an analytical solution and so numerical methods must be employed. Here we use a method based on the Nikaido-Isoda function and a relaxation algorithm (hence the name: NIRA).

3.2.1 The Nikaido-Isoda function

This function is a cornerstone of the NIRA technique for solving games for their CCE. It transforms the complex process of solving a (constrained) game into a far simpler (constrained) optimisation problem.

In the remainder of this section (*i.e.*, section 3), we are indexing a player's action, payoff, weight, *etc.* by the sole index f. If the players' responsibilities are distributed non-equally among periods t, then all these variables should be indexed by f *and* t. We will do so in sections 4 and 5 where the game solutions are computed and interpreted. Here, however, for clarity of notation and interpretations of the Nikaido-Isoda function we use the single index.

Definition 3.1. *Let ϕ_f be the payoff function for player f, X a collective strategy set as before and $r_f > 0$ be a given weighting*[10] *of player f. The* Nikaido-Isoda *function $\Psi : X \times X \to I\!R$ is defined as*

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{f=1}^{F} r_f [\phi_f(y_f|\mathbf{x}) - \phi_f(\mathbf{x})] \tag{6}$$

Result 3.1. *See [21].*

$$\Psi(\mathbf{x}, \mathbf{x}) \equiv 0 \qquad\qquad \mathbf{x} \in X. \tag{7}$$

[9]In our study, the payoff function $\phi_f(\cdot)$ corresponds to the expression under the maximisation operator in (1).

[10]The weights can be viewed as a political instrument the regulator might use to distribute the responsibility for the joint constraints' satisfaction, among the generators and periods. If the players' responsibilities are distributed in periods t non-equally, then the weights, actions, payoffs *etc.* become r_f^t, x_f^t, ϕ_f^t, respectively.

Each summand from the Nikaido-Isoda function can be thought of as the improvement in payoff a player will receive by changing his action from x_f to y_f while all other players continue to play according to $\mathbf{x}$. Therefore, the function represents the sum of these improvements in payoff. Note that the *maximum* value this function can take, for a given $\mathbf{x}$, is always nonnegative, owing to Result 3.1 above. The function is everywhere non-positive when either $\mathbf{x}$ or $\mathbf{y}$ is a Nash equilibrium point, since in an equilibrium situation no player can make any improvement to their payoff. Consequently, each summand in this case can be at most zero at the Nash equilibrium point [13].

We observe that the "sum of improvements" in Ψ depends on the weighting vector $\mathbf{r} = (r_f)_{f\in F}$. Consequently, a manifold of equilibria indexed by $\mathbf{r}$ is expected to exist. However, for a *given* $\mathbf{r}$ and diagonal strict concavity of $\sum_{f\in F} r_f \phi_f(x_f)$, uniqueness of equilibrium $\mathbf{x}^*$ is guaranteed, see [19] and [7]. Also, notice that according to Theorem 3.1 (the convergence theorem), if the assumptions are fulfilled, then NIRA converges to the unique equilibrium, for the value of $\mathbf{r}$ that was used in the definition of Ψ.

When the Nikaido-Isoda function cannot be made (significantly) positive for a given $\mathbf{y}$, we have (approximately) reached the Nash equilibrium point. This observation is used to construct a termination condition for the relaxation algorithm, which is used to min-maximise Ψ. An ε is chosen such that, when

$$\max_{\mathbf{y}\in\mathbb{R}^m} \Psi(\mathbf{x}^s, \mathbf{y}) < \varepsilon, \tag{8}$$

(where $\mathbf{x}^s$ is the s-th iteration approximation of x^*) the Nash equilibrium would be achieved to a sufficient degree of precision [13].

The Nikaido-Isoda function is used to construct the optimum response function. This function is similar to the best response function in standard non-cooperative game theory. It defines each player's optimal action to maximise his payoff given what the other players have chosen. The vector $Z(\mathbf{x})$ gives the 'best move' of each player when faced with the collective action $\mathbf{x}$. It is at this point that the coupled constraints are introduced into the optimisation problem. The maximisation of the Nikaido-Isoda function in equation (9) is performed subject to the constraints on the players' actions.

Definition 3.2. *The* optimum response function *at point* $\mathbf{x}$ *is*

$$Z(\mathbf{x}) \in \arg\max_{\mathbf{y}\in X} \Psi(\mathbf{x}, \mathbf{y}). \tag{9}$$

3.2.2 The relaxation algorithm

The relaxation algorithm iterates the function Ψ to find the Nash equilibrium of a game. It starts with an initial estimate of the Nash equilibrium and iterates from that point towards $Z(\mathbf{x})$ until no more improvement is possible. At such a point every player is playing their optimum response to every other player's action and the Nash equilibrium is reached. The relaxation algorithm, when $Z(\mathbf{x})$ is single-valued, is

$$\mathbf{x}^{s+1} = (1-\alpha_s)\mathbf{x}^s + \alpha_s Z(\mathbf{x}^s) \qquad \begin{array}{l} 0 < \alpha_s \leq 1 \\ s = 0, 1, 2, \ldots \end{array} \tag{10}$$

From the initial estimate, an iterate step $s+1$ is constructed by a weighted average of the players' improvement point $Z(\mathbf{x}^s)$ and the current action point $\mathbf{x}^s$. Given concavity assumptions explained in section 3.3, this averaging ensures convergence (see [21], [13]) to the Nash equilibrium by the algorithm. By taking a sufficient number of iterations of the algorithm, the Nash equilibrium $\mathbf{x}^*$ can be determined with a specified precision.

3.3 Existence and uniqueness of equilibrium points

It is one thing to know that one has a method to solve games with constraints but, before proceeding, one needs to establish that the game has an equilibrium at all. Furthermore, since the NIRA algorithm converges to a single equilibrium point it would be nice if that equilibrium could be shown to be unique. The conditions for existence and uniqueness for games with coupled constraints are more intricate than those for games with uncoupled constraints. It is known that every concave n-person game with uncoupled constraints has an equilibrium point [19]. The equivalent definition for a game with coupled constraints relies upon the notion of a weakly convex-concave function.

A weakly convex-concave function is a bivariate function that exhibits weak convexity in its first argument and weak concavity in its second argument. The next three definitions (see [16] or [21]) formalise this notion.[11] As Theorem 3.1 (the convergence theorem) will document, weak convex-concavity of a function is a crucial assumption needed for convergence of a relaxation algorithm to a coupled constraints equilibrium.

Let X be a convex closed subset of the Euclidean space $\mathbb{R}^m$ and f a continuous function $f : X \to \mathbb{R}$.

Definition 3.3. *A function of one argument $f(\mathbf{x})$ is* weakly convex *on X if there exists a function $r(\mathbf{x},\mathbf{y})$ such that $\forall \mathbf{x},\mathbf{y} \in X$*

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \geq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) r(\mathbf{x},\mathbf{y}) \tag{11}$$
$$0 \leq \alpha \leq 1, \text{ and } \frac{r(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X.$$

Definition 3.4. *A function of one argument $f(\mathbf{x})$ is* weakly concave *on X if there exists a function $\mu(\mathbf{x},\mathbf{y})$ such that, $\forall \mathbf{x},\mathbf{y} \in X$*

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \leq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x},\mathbf{y}) \tag{12}$$
$$0 \leq \alpha \leq 1, \text{ and } \frac{\mu(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{as} \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X.$$

Example: *The convex function $f(x) = x^2$ is weakly concave (see [13]) but the convex function $f(x) = |x|$ is not.*

Now take a bivariate function $\Psi : X \times X \to \mathbb{R}$ defined on a product $X \times X$, where X is a convex closed subset of the Euclidean space $\mathbb{R}^m$.

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[11] Recall the following elementary definition: a function is "just" *convex* $\iff$

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \ \geq\ f\Big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\Big), \quad \alpha \in [0,1].$$

Definition 3.5. *A function of two vector arguments, $\Psi(\mathbf{x}, \mathbf{y})$ is referred to as* weakly convex-concave *if it satisfies weak convexity with respect to its first argument and weak concavity with respect to its second argument.*

The functions $r(\mathbf{x}, \mathbf{y}; \mathbf{z})$ and $\mu(\mathbf{x}, \mathbf{y}; \mathbf{z})$ were introduced with the concept of weak convex-concavity and are called the *residual terms.* Notice that smoothness of $\Psi(\mathbf{z}, \mathbf{y})$ is not required. However, if $\Psi(\mathbf{x}, \mathbf{y})$ is twice continuously differentiable with respect to both arguments on $X \times X$, the residual terms satisfy (see [13])

$$r(\mathbf{x}, \mathbf{y}; \mathbf{y}) = \tfrac{1}{2}\langle A(\mathbf{x}, \mathbf{x})(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}\rangle + o_1(\|\mathbf{x} - \mathbf{y}\|^2) \tag{13}$$

and

$$\mu(\mathbf{y}, \mathbf{x}; \mathbf{x}) = \tfrac{1}{2}\langle B(\mathbf{x}, \mathbf{x})(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}\rangle + o_2(\|\mathbf{x} - \mathbf{y}\|^2) \tag{14}$$

where $A(\mathbf{x}, \mathbf{x}) = \Psi_{\mathbf{xx}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the first argument and $B(\mathbf{x}, \mathbf{x}) = \Psi_{\mathbf{yy}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$.

To prove the inequality of condition (5) of Theorem 3.1 (the convergence theorem, below) under the assumption that $\Psi(\mathbf{x}, \mathbf{y})$ is twice continuously differentiable, it suffices to show that

$$Q(\mathbf{x}, \mathbf{x}) = A(\mathbf{x}, \mathbf{x}) - B(\mathbf{x}, \mathbf{x}) \tag{15}$$

is strictly positive definite.

Theorem 3.1 (Convergence theorem)**.** *There exists a unique normalised Nash equilibrium point to which the algorithm* (10) *converges if:*

1. *X is a convex, compact subset of $\mathbb{R}^m$,*
2. *the Nikaido-Isoda function $\Psi : X \times X \to \mathbb{R}$ is a weakly convex-concave function and $\Psi(\mathbf{x}, \mathbf{x}) = 0$ for $\mathbf{x} \in X$,*
3. *the optimum response function $Z(\mathbf{x})$ is single valued and continuous on X,*
4. *the residual term $r(\mathbf{x}, \mathbf{y}; \mathbf{z})$ is uniformly continuous on X w.r.t. $\mathbf{z}$ for all $\mathbf{x}, \mathbf{y} \in X$,*
5. *the residual terms satisfy*
$$r(\mathbf{x}, \mathbf{y}; \mathbf{y}) - \mu(\mathbf{y}, \mathbf{x}; \mathbf{x}) \geq \beta(\|\mathbf{x} - \mathbf{y}\|), \qquad \mathbf{x}, \mathbf{y} \in X \tag{16}$$
*where $\beta(0) = 0$ and β is a strictly increasing function (*i.e.*, $\beta(t_2) > \beta(t_1)$ if $t_2 > t_1$),*
6. *the relaxation parameters α_s satisfy*
   - *either (non-optimised step)*

     (a) $\alpha_s > 0$,

(b) $\sum_{s=0}^{\infty} \alpha_s = \infty$,

(c) $\alpha_s \to 0$ *as* $s \to \infty$.

- *or (optimised step)*

$$\alpha_s = \arg \min_{\alpha \in [0,1)} \left\{ \max_{\mathbf{y} \in X} \Psi(\mathbf{x}^{(s+1)}(\alpha), \mathbf{y}) \right\}. \tag{17}$$

Proof. See [13] for a proof. □

3.4 Enforcement through taxation

Once a CCE, x^*, has been computed it is possible to create an unconstrained game which has x^* as its solution by a simple modification to the players' payoff functions. For example, a regulator may compute that x^* is the CCE of a game involving the desired constraints on agents' behaviour. He may then wish to induce the players to arrive at this point through a scheme of taxation that modifies their payoff functions. This can be achieved by the use of penalty functions that punish players for breaching the coupled constraints.

Penalty functions are weighted by the Lagrange multipliers obtained from the constrained game. For each constraint, players are taxed according to the function

$$T_{\ell,f}(\lambda, r_f, \mathbf{x}) = \frac{\lambda_\ell}{r_f} \max(0, Q_\ell(\mathbf{x}) - \overline{Q}_\ell) \tag{18}$$

where λ_ℓ is the Lagrange multiplier associated with the ℓth constraint $Q_\ell(\mathbf{x})$ may be the amount of pollution as described by the left hand side of (4), or of the transmitted power as in (3), while $\overline{Q}_\ell$, $\ell = 1, 2, \ldots L$ denote the corresponding limits (L is the total number of constraints); $\mathbf{x}$ is the vector of players' actions, r_f is player f's weight that defines their responsibility for the constraints' satisfaction.

If the weights $\mathbf{r}$ were identical $[1\,, 1\,, \ldots 1]$ (see [3]) then the penalty term for constraint ℓ is the same for each player f

$$T_{\ell,f}(\lambda, 1, \mathbf{x}) = \lambda_\ell \max(0, Q_\ell(\mathbf{x}) - \overline{Q}_\ell)\,.$$

Hence, if the weight for player f is for example $r_f > 1$ and the weights for the other players were $1, 1, \ldots 1$, then the responsibility of player f for the constraints' satisfaction is lessened. Obviously, if the players' responsibilities are distributed in periods t non-equally, then the weights, as well as the other variables, need the other index t.

The players' payoff functions, so modified, will be

$$\underline{\phi}_f(\mathbf{x}) = R_f(\mathbf{x}) - C_f(\mathbf{x}) - \sum_\ell T_{\ell f}(\lambda, \mathbf{r}, \mathbf{x}) \tag{19}$$

where R_f and C_f are firm f's revenue and cost functions respectively. Notice that under this taxation scheme the penalties remain "nominal" (*i.e.*, zero) if all constraints are satisfied.

The Nash equilibrium of the new unconstrained game with payoff functions $\overline{\phi}$ is implicitly defined by the equation

$$\underline{\phi}(\mathbf{x}^{**}) = \max_{y_f \in \mathbb{R}^+} \underline{\phi}(y_f|\mathbf{x}^{**}) \qquad \forall\, f, \tag{20}$$

(compare with equation (5)). For the setup of the problem considered in this paper $x^* = x^{**}$. That is, the CCE is equal to the unconstrained equilibrium with penalty functions for breaches of the constraints, weighted by the Lagrange multipliers (see [13], [10] and [11] for a more detailed discussion).

4 Case study

4.1 Without coupled constraints

The example is taken from [2] (originally, from [8]). Numerical data for the general formulation of the problem is as follows. The period of study is split into two sub-periods, weekdays and weekends, for which the demands for electricity are different. The periods account for $5 \cdot 24$ h=120 h and $2 \cdot 24$ h=48 h per week, or in annual terms to 5/7 and 2/7 of the $365 \cdot 24 = 8,760$ h of an entire year *i.e.*, 6,257 h and 2,503 h , respectively. These periods correspond to the two parts, into which the load duration curve is divided. We assume that every hour within each period is identical in terms of demand for energy, sales, power generations, *etc.*

There are three nodes, $i = 1,\ 2,\ 3$, each of which has customers. Generation occurs only at nodes 1 and 2 and each pair of nodes is connected by a single transmission line. The demand functions (per hour) are: in period 1, $p_i^1(\sigma_i^1) = 40 - 0.08\sigma_i^1$, for nodes $i = 1,\ 2$ and $p_3^1(\sigma_3^1) = 32 - 0.0516\sigma_3^1$ \$/MWh for node 3; in period 2, $p_i^2(\sigma_i^2) = 30 - 0.06\sigma_i^2$, for nodes $i = 1,\ 2$ and $p_3^2(\sigma_3^2) = 24 - 0.0387\sigma_3^2$ \$/MWh for node 3. Thus, the demand is more elastic at the demand-only node 3. Firm's 1 generator is placed at $i = 1$ and firm's 2 at $i = 2$. Since each firm has only one generator we drop the i and g subscripts for brevity (*e.g.*, P_{fgi}^t becomes P_f^t). Both generators have unlimited capacity and constant marginal costs $\mathbb{MC}_1 = \dfrac{dC(P_1^1)}{dP_1^1} = \dfrac{dC(P_1^2)}{dP_1^2} = 15$ for firm 1 and $\mathbb{MC}_2 = \dfrac{dC(P_2^1)}{dP_2^1} = \dfrac{dC(P_2^2)}{dP_2^2} = 20$ for firm 2. The marginal costs are measured in \$/MWh. The three lines have equal impedances of 0.2 p.u. The slack node is node 3.

Bearing in mind the general formulation of the *generation game* defined by the payoffs (1) and constraints (including coupled) (2)–(4) and the numerical values for the problem parameters as above, both firms solve the following optimisation problems:

Firm 1:

$$\begin{aligned} \max\Big\{&[40 - 0.08\sigma_1^1]s_{11}^1 + [40 - 0.08\sigma_2^1]s_{12}^1 + [32 - 0.0516\sigma_3^1]s_{13}^1 - 15P_1^1 \cdot 6,257 + \\ &[30 - 0.06\sigma_1^2]s_{11}^2 + [30 - 0.06\sigma_2^2]s_{12}^2 + [24 - 0.0387\sigma_3^2]s_{13}^2 - 15P_1^2 \cdot 2,503\Big\} \end{aligned} \tag{21}$$

Firm 2:

$$\begin{aligned} \max\Big\{&[40 - 0.08\sigma_1^1]s_{21}^1 + [40 - 0.08\sigma_2^1]s_{22}^1 + [32 - 0.0516\sigma_3^1]s_{23}^1 - 20P_2^1 \cdot 6,257 + \\ &[30 - 0.06\sigma_1^2]s_{21}^2 + [30 - 0.06\sigma_2^2]s_{22}^2 + [24 - 0.0387\sigma_3^2]s_{23}^2 - 20P_2^2 \cdot 2,503\Big\} \end{aligned} \tag{22}$$

$$
\begin{aligned}
P_1^1\Delta^1 &= s_{11}^1 + s_{12}^1 + s_{13}^1, && \text{(23a)}\\
P_1^2\Delta^1 &= s_{11}^2 + s_{12}^2 + s_{13}^2, && \text{(23b)}\\
P_2^1\Delta^2 &= s_{21}^1 + s_{22}^1 + s_{23}^1, && \text{(23c)}\\
P_2^2\Delta^2 &= s_{21}^2 + s_{22}^2 + s_{23}^2, && \text{(23d)}\\
q_1^1 &= s_{11}^1 + s_{21}^1, && \text{(23e)}\\
q_1^2 &= s_{11}^2 + s_{21}^2, && \text{(23f)}\\
q_2^1 &= s_{12}^1 + s_{22}^1, && \text{(23g)}\\
q_2^2 &= s_{12}^2 + s_{22}^2, && \text{(23h)}\\
q_3^1 &= s_{13}^1 + s_{23}^1, && \text{(23i)}\\
q_3^2 &= s_{13}^2 + s_{23}^2, && \text{(23j)}\\
&\text{all } s_{11}^1, s_{11}^2, s_{12}^1, s_{12}^2, s_{13}^1, s_{13}^2, s_{21}^1, s_{21}^2, s_{22}^1, s_{22}^2, s_{23}^1, s_{23}^2,\\
&\overline{P}_{1\to2}^1, \overline{P}_{1\to2}^2, \overline{P}_{1\to3}^1, \overline{P}_{1\to3}^2, \overline{P}_{2\to3}^1, \overline{P}_{2\to3}^2, \text{ nonnegative,}
\end{aligned}
$$

where the decision variables of the generators (firms) are: $s_{11}^1, s_{11}^2, s_{12}^1, s_{12}^2, s_{13}^1$ and s_{13}^2 for the first generator, and $s_{21}^1, s_{21}^2, s_{22}^1, s_{22}^2, s_{23}^1$ and s_{23}^2 for the second generator. As in section 2.2, $\sigma_i^t = q_i^t/\Delta^t$.

The remaining variables are dependent on the decision variables. If the problem involves constraints then part of the solution will constitute the Lagrange multipliers that a regional regulator will be able to use to enforce the equilibrium (see section 3.4).

4.2 Transmission line constraints

A constraint on transmission line capacity per period of the load duration curve is imposed as described in equation (3).[12] The equations of the constraints in this example are

$$
\left| \left[B_d \cdot A^T \cdot B^{-1} \right] \cdot \begin{pmatrix} P_1^1 - q_1^1/\Delta^1 \\ P_2^1 - q_2^1/\Delta^1 \end{pmatrix} \right| \le \begin{pmatrix} \overline{P}_{1\to2}^1 \\ \overline{P}_{1\to3}^1 \\ \overline{P}_{2\to3}^1 \end{pmatrix}. \tag{25}
$$

$$
\left| \left[B_d \cdot A^T \cdot B^{-1} \right] \cdot \begin{pmatrix} P_1^2 - q_1^2/\Delta^2 \\ P_2^2 - q_2^2/\Delta^2 \end{pmatrix} \right| \le \begin{pmatrix} \overline{P}_{1\to2}^2 \\ \overline{P}_{1\to3}^2 \\ \overline{P}_{2\to3}^2 \end{pmatrix}. \tag{26}
$$

[12] For example, the first row of $P_{i\to j}^1$ is

$$
P_{1\to2}^1 = 0.66\, s_{12}^1/\Delta^1 + 0.33\, s_{13}^1/\Delta^1 - 0.66\, s_{21}^1/\Delta^1 - 0.33\, s_{23}^1/\Delta^1. \tag{24}
$$

This indicates that the flow along the line from node 1 to node 2 in period 1 depends not only upon the quantity that is sold to nodes 1 and 2 but also upon the quantity of electricity that is sold to node 3.

The values of the transmission line constraints are as follows:

$$B_d = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 10 & -5 \\ -5 & 10 \end{pmatrix}. \tag{27}$$

Note that the first row of matrix A (whose dimension is: 2 nodes times 3 lines) expresses that node 1 is the starting node of lines 1–2 and 1–3, and the second row means that node 2 is the ending node of line 1–2 and the starting node of line 2–3. Node 3 is the slack node for which there are no calculations, since it is the reference node.

The diagonal terms of B are computed as follows: b_{11} is the sum of the two line susceptances connected to node 1, likewise for the other diagonal term corresponding to node 2. The off-diagonal terms are the susceptances of the lines 1–2 and 2–1 (which is the same line), respectively.

Substituting the matrices in (27) into equations (25) and (26), and choosing line limits $(25, 200, 200)'$, gives this numerical expression for the line constraints in periods 1 and 2:

$$\left| \begin{pmatrix} 0.33 & -0.33 \\ 0.66 & 0.33 \\ 0.33 & 0.66 \end{pmatrix} \cdot \begin{pmatrix} P_1^1 - q_1^1/\Delta^1 \\ P_2^1 - q_2^1/\Delta^1 \end{pmatrix} \right| \leq \begin{pmatrix} 25 \\ 200 \\ 200 \end{pmatrix}. \tag{28}$$

$$\left| \begin{pmatrix} 0.33 & -0.33 \\ 0.66 & 0.33 \\ 0.33 & 0.66 \end{pmatrix} \cdot \begin{pmatrix} P_1^2 - q_1^2/\Delta^2 \\ P_2^2 - q_2^2/\Delta^2 \end{pmatrix} \right| \leq \begin{pmatrix} 25 \\ 200 \\ 200 \end{pmatrix}. \tag{29}$$

$\overline{P}^t_{1\to3}$ and $\overline{P}^t_{2\to3}$ are set arbitrarily large such that they never bind. Because of this, our problem can be considered to have only one coupled constraint on $P^t_{1\to2}$. Lagrange multipliers are only reported for this active constraint in the results below.

4.3 Environmental constraints

In this case study, an emission constraint is added to the problem formulation. As a result, the problem is set as in (21)–(23), where both firms have the same optimization functions as before, but, in this case, a new environmental constraint is added to the constraint set (23), so that:

$$\begin{aligned} &6,257 \cdot [(20 - 0.4 \cdot P_1^1 + 0.004 \cdot (P_1^1)^2) + (22 - 0.3 \cdot P_2^1 + 0.005 \cdot (P_2^1)^2)] + \\ &2,503 \cdot [(20 - 0.4 \cdot P_1^2 + 0.004 \cdot (P_1^2)^2) + (22 - 0.3 \cdot P_2^2 + 0.005 \cdot (P_2^2)^2)] \leq 2,190,000, \end{aligned} \tag{30}$$

where the maximum allowed emission is 2,190,000 lb. Note[13]. that (30) is applied to periods 1 and 2 simultaneously. Consequently, this constraint has a time coupling effect.

The annual emission limit assumes a constant hourly value of 250 lb/h that, multiplied by 8,760 hours, results in 2,190,000 lb per year.

[13] We turn the reader's attention to the fact that the function in expression (30) is quadratic so, the second superindex means "square".

4.4 What kind of stylised situation reflects the model

Our first generator is economically efficient and has good filters. The other generator is "outdated" and does not have good filters. The transmission line between the generators is due for an upgrade while the other lines are recently rebuilt and have a high transmission capacity.

As judged by the nodal demand laws, the first two nodes represent localities with a mixture of industrial and domestic customers. The third node corresponds to a township or a conglomerate of rural customers whose demand for electricity is lower than in the first two localities.

As said earlier, the demand functions change depending on the period. It is assumed that the demand in weekdays (first period) is higher than the demand on weekends (second period).

The local regulator is concerned about pollution levels at some critical location (like a National Park) where a monitor is installed. The monitor is capable of detecting if the total annual pollution exceeds 2,190,000 lb, see condition (30). Presumably, the coefficients of the pollution function in this equation have been calibrated to reflect the emissions' transportation process from each generator to the "critical" location.

The local environmental lobby group has managed to legislate that the limit of $K = 2,190$ klbs *per* year, measured at the "critical" location, has to be obeyed. The local regulator considers an introduction of a pollution charge to be paid by both generators should a violation of this constraint be detected.

The local regulator is interested to know what the generators' reactions could be should the pollution charge become a reality. In particular, the regulator would like to know what energy prices can be expected after the introduction, whether the energy supply might be affected and in what way the region's welfare may be influenced. The results presented below provide some answers to these questions.

5 The first round's results

5.1 Sharing the constraints' burden in solidarity

We anticipate that, at realising what impact the constraints on the local economy might have ("first round" - this section), the regulator will try and test some measures to "save" the consumer surplus ("second round" - section 6).

As said earlier[14], the generation game (21), (22), (23) (with constraints (28),(29) and (30), if appropriate) satisfies the hypotheses of the convergence theorem (page 10) of the NIRA method. Consequently, we have used NIRA (see section 3.2) to obtain the game's solutions[15] reported below. In the "first round", we set all weights $r_f^t \equiv 1$ (commented on in footnote 10, page 7).

[14] See [2].

[15] We remind the reader that a solution (equilibrium) is a combination of the decision variables' values such that the constraints are satisfied and no player is able to improve his payoff by an unilateral move.

5.2 Base case

The results of the application of NIRA to the game defined by (21)–(23) are shown in tables I and II. These and subsequent tables have three (or two) rows of results: all first row's numbers concern the first period, the second row's refer to the second period. The third row (if appropriate) provides the sums for both periods.

The energy demanded per node for the first period is: 1,173.18 GWh, 1,173.18 GWh, and 1,172.18 GWh and for the second period, 347.64 GWh, 347.64 GWh, and 291.05 GWh. As expected, the "weekend" demand is lower than the weekdays'; also, the third node ("rural") consumes less energy than the other nodes.

Prices at the nodes, according to the linear demand functions, are: 25, 25, and 22.3 $/MWh for the first period and 21.66, 21.66 and 19.5 $MWh for the second period.

The flows' absolute values per hour through the lines are: 73.96 MW, 130.65 MW, and 56.7 MW for lines 1–2, 1–3 and 2–3, in the first period and 94.31 MW, 105.29 MW, and 10.98 MW for lines 1–2, 1–3 and 2–3 in the second period.

Profits for the first period for firms 1 and 2 are 22.16 M$ and 4.57 M$, respectively; for the second period, they are 5.02 M$ and 232 k$, respectively. As expected, the first generator produces substantially more than the second. Total pollution is 4,627 klb. As no constraints are active the Lagrange multipliers for the line and environmental constraints are: $\lambda^1_{L1} = \lambda^2_{L1} = \lambda_E = 0$ (notice that we do not index λ_E with time because this Lagrange multiplier corresponds to the intertemporary constraint (30)).

Table I: Sales, generation and total emissions in the base case

Sales by Firm 1 (1000MWh)			Sales by Firm 2 (1000MWh)			Generation by Firms (1000MWh)		Emissions by Firms (1000lb)	
s^t_{11}	s^t_{12}	s^t_{13}	s^t_{21}	s^t_{22}	s^t_{23}	$P^t_1\Delta^t$	$P^t_2\Delta^t$	E^t_1	E^t_2
782.12	782.12	889.24	391.06	391.06	282.94	2453.5	1065.1	2992	724.6
278.11	278.11	291.05	69.53	69.53	0	847.3	139.1	858.4	52
1060.24	1060.24	1180.3	460.6	460.6	282.94	3300.8	1204.2	3850.4	776.6

Table II: Economics of generation in the base case per hour per period

Demand (MW)			Prices (\$/MWh)			Profits (\$/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
187.5	187.5	187.34	25	25	22.33	3542.2	730.51
138.89	138.89	116.28	21.66	21.66	19.5	2004.8	92.7
Flows for lines (MW)			Lagrange multipliers		Emission levels (lb/h)		
$1-2$	$1-3$	$2-3$	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
73.96	130.65	56.69	-	-	478.18	115.81	593.99
94.31	105.29	10.98	-	-	342.93	20.76	363.7

5.3 Generation under a transmission constraint

For this example, a limit of 25 MW in the transmission capacity of the line that connects nodes 1 and 2 is imposed. As a result, the problem is set as in section 5.2 above with the addition of the constraints described in equations (25) and (26).

This generation game is a game with coupled constraints, among two non identical players (player 1 produces electricity more cheaply than player 2). From the results on flows shown in table II we expect the constraints (28), (29) will be binding.

The results of the relaxation algorithm are shown in tables III and IV. The energy demanded per node for the first period is: 1,245.8 GWh, 1,100.5 GWh, and 1,172.2 GWh and for the second period, 391.3 GWh, 303.9 GWh, and 280.3 GWh.

Table III: Generation and sales with a line flow limit of 25 MW in line 1–2

Sales by Firm 1 (1000MWh)			Sales by Firm 2 (1000MWh)			Generation by Firms (1000MWh)		Emissions by Firms (1000lb)	
s_{11}^t	s_{12}^t	s_{13}^t	s_{21}^t	s_{22}^t	s_{23}^t	$P_1^t\Delta^t$	$P_2^t\Delta^t$	E_1^t	E_2^t
709.47	636.82	720.28	536.37	463.72	451.9	2066.6	1452	2028.7	1386.8
234.41	190.72	200.2	156.92	113.22	80.06	625.3	350.2	424.8	195
943.89	827.54	920.48	693.29	576.94	531.96	2691.9	1802.2	2453.5	1581.8

Prices at the nodes are: 24.1, 25.9, and 22.3 \$/MWh for the first period and 20.6, 22.7 and 19.6 \$/MWh for the second period. The flows' absolute values per hour through the lines are: 25 MW, 106.17 MW, and 81.17 MW for lines 1–2, 1–3 and 2–3, in the first period and 25 MW, 68.5 MW, and 43.5 MW for lines 1–2, 1–3 and 2–3 in the second period. We notice that the line constraint is binding and that most prices are higher for this constrained generation case than for the previous one when no constraint was imposed.

Profits for the first period for firms 1 and 2 are 18.67 M\$ and 5.98 M\$, respectively; for the second period, they are 3.72 M\$ and 378 k\$, respectively. Total pollution is 4,035 klb. The Lagrange multipliers for the line constraints are:

$\lambda_{L1}^{1} = 26.16$; $\lambda_{L1}^{2} = 11.8$; the environmental constraint is not imposed hence. $\lambda_E = 0$. Interestingly, the total energy generation is comparable between this and this previous cases; however, firm 2 produces now more than before, hence its marked share has increased.

Table IV: Economics of generation per hour per period with a line flow limit of 25 MW in line 1–2

Demand (MW)			Prices ($/MWh)			Profits ($/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
199.11	175.89	187.34	24.07	25.93	22.33	2985.04	956.90
156.35	121.43	111.97	20.62	22.71	19.67	1487.42	151.02
Flows for lines (MW)			Lagrange multipliers		Emission levels (lb/h)		
$1-2$	$1-3$	$2-3$	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
25	106.17	81.17	26.16	-	324.23	221.64	545.87
25	68.49	43.49	11.8	-	169.73	77.91	247.64

5.4 Generation under an environmental constraint

The regulator will also be interested to see in what way the generators' behaviour would change if the environmental constraint (30) was introduced (and the transmission line upgraded so that the line constraint became non-binding). The results obtained are shown in tables V and VI.

Table V: Generation and sales with an emission constraint

Sales by Firm 1 (1000MWh)			Sales by Firm 2 (1000MWh)			Generation by Firms (1000MWh)		Emissions by Firms (1000lb)	
s_{11}^t	s_{12}^t	s_{13}^t	s_{21}^t	s_{22}^t	s_{23}^t	$P_1^t\Delta^t$	$P_2^t\Delta^t$	E_1^t	E_2^t
574.62	574.62	567.53	347.34	347.34	215.16	1716.8	909.8	1322.6	526.2
182.92	182.92	154.24	87.86	87.86	6.86	520.1	182.6	274.3	66.9
757.55	757.55	721.78	435.2	435.2	222.02	2236.9	1092.4	1596.9	593.1

The energy demanded per node for the first period is: 921.9 GWh, 921.9 GWh, and 782.7 GWh and for the second period, 270.7 GWh, 270.7 GWh, and 161.1 GWh. Prices at the nodes are: 28.2, 28.2 and 25.5 $/MWh for the first period and 23.5, 23.5 and 21.5 $MWh for the second period. We notice that the prices rose even higher than under the transmission constraint.

The flows' absolute values per hour through the lines are: 42.98 MW, 84.04 MW, and 41.05 MW for lines 1–2, 1–3 and 2–3, in the first period and 44.94 MW, 54.66 MW, and 9.71 MW for lines 1–2, 1–3 and 2–3 in the second period.

Profits for the first period for firms 1 and 2 are 21.17 M$ and 6.89 M$, respectively; for the second period, they are 4.11 M$ and 627 k$, respectively. Total

Table VI: Economics of generation per hour per period with an emission constraint

Demand (MW)			Prices ($/MWh)			Profits ($/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
147.35	147.35	125.09	28.21	28.21	25.54	3383.25	1102.44
108.18	108.18	64.36	23.51	23.51	21.51	1644.83	250.5
Flows for lines (MW)			**Lagrange multipliers**		**Emission levels (lb/h)**		
1−2	1 − 3	2 − 3	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
42.99	84.04	41.05	-	-	211.38	84.1	295.48
44.95	54.66	9.71	-	0.0033	109.58	26.72	136.3

pollution is 2,190.0 klb. The Lagrange multipliers for the environmental constraint is: $\lambda_E = 0.0033$. Also in this constrained generation game, firm 2's production levels increase above the no constrained game's; as a result its market share increases.

5.5 Generation under transmission and environmental constraints

In this case study, both the 25 MW thermal limit of line 1–2 (equations (25) and (26)) and the emission constraints (equation (30)) are added to the basic problem formulation.

The results obtained are shown in tables VII and VIII. The energy demanded per node for the first period is: 989.1 GWh, 883 GWh, and 804.6 GWh and for the second period, 301.1 GWh, 247.4 GWh, and 166.5 GWh.

Prices at the nodes, according to the linear demand functions, are: 27.35, 28.71 and 25.36 $/MWh for the first period and 22.78, 24.07 and 21.43 $MWh for the second period. The flows' absolute values per hour through the lines are: 25 MW, 76.79 MW, and 51.79 MW for lines 1–2, 1–3 and 2–3, in the first period and 25 MW, 45.76 MW, and 20.76 MW for lines 1–2, 1–3 and 2–3 in the second period.

Table VII: Generation and sales with a line flow limit of 25 MW in line 1–2 and an emission constraint

Sales by Firm 1 (1000MWh)			Sales by Firm 2 (1000MWh)			Generation by Firms (1000MWh)		Emissions by Firms (1000lb)	
s_{11}^t	s_{12}^t	s_{13}^t	s_{21}^t	s_{22}^t	s_{23}^t	$P_1^t\Delta^t$	$P_2^t\Delta^t$	E_1^t	E_2^t
575.62	522.54	527.93	413.54	360.46	276.65	1626.1	1050.7	1165.1	704.6
184.59	157.69	135.98	116.56	89.67	30.52	478.3	236.8	224.3	96
760.21	680.24	663.92	530.11	450.14	307.17	2104.4	1287.5	1389.4	800.6

Profits for the first period for firms 1 and 2 are 19.74 M$ and 7.66 M$, respectively; for the second period, they are 3.74 M$ and 733 k$, respectively. Total pollution is 2,190 klb. The Lagrange multipliers for the line and environmental constraints are: $\lambda_{L1}^1 = 19.11$; $\lambda_{L1}^2 = 7.26; \lambda_E = 0.003$.

Table VIII: Economics of generation per hour per period with a line flow limit of 25 MW in line 1–2 and an emission constraint

Demand (MW)			Prices (\$/MWh)			Profits (\$/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
158.09	141.12	128.59	27.35	28.71	25.36	3155.98	1225.02
120.31	98.82	66.52	22.78	24.07	21.43	1494.2	292.85
Flows for lines (MW)			Lagrange multipliers		Emission levels (lb/h)		
$1-2$	$1-3$	$2-3$	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
25	76.79	51.79	19.11	-	186.21	112.6	298.81
25	45.76	20.76	7.26	0.003	89.61	38.36	127.97

5.6 First round's discussion

We notice that the imposition of constraints favours the second generator in that its share of production increases, see figure 1. The symbols in the figure correspond to the above four cases as follows:

1. *square* – no constraints (section 5.2);
2. *diamond* – transmission line constraint (section 5.3);
3. *pentagram* – environmental constraint (section 5.4);
4. *hexagram* – the two constraints jointly (section 5.5).

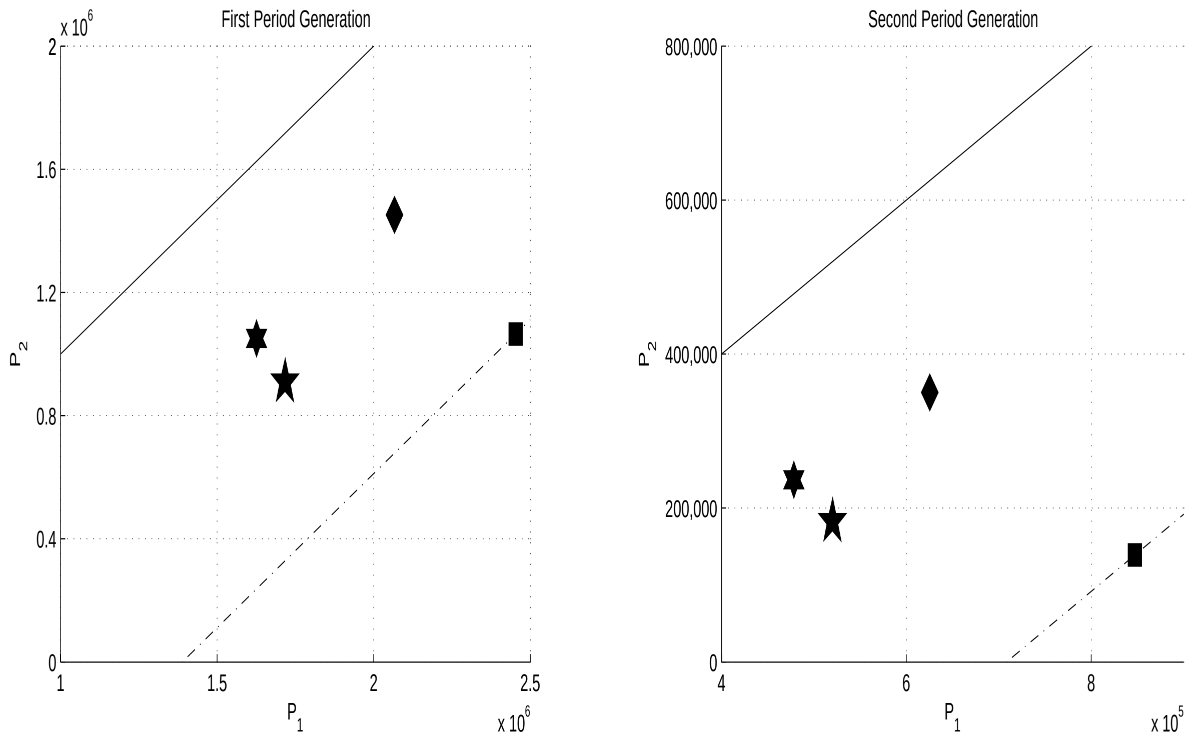

Figure 1: Generations (in MWh) in periods 1 and 2.

All lines in figure 1 are angled at 45^o. The solid ones go through the origin. If the generators were identical, the constraints would affect each them in the same way and the generation points would be on this line. Obviously, the generators in the case study are not identical. However, imposition of a constraint blurs the difference. This is visible from the diminishing distances of the generation points from the 45^o line, after a constraint imposition.

The dash-dotted lines pass through the unconstrained generation point (*square*). They are drawn to help assess the distance of each generation point from the 45^o line.

The overall result is that the constrained generation games result in the firm 2's (the less efficient one) gains of the market. This can be qualified as market distortion. However, when a transmission constraint is the only model restriction, the distortion concerns the generators rather than consumers. In case of the imposition of an environmental constraint (sole or joint with a transmission constraint) the prices rise substantially and the consumers surplus is diminished. We will comment on this issue later in section 7.1.

We conjecture that the market distortion due to constraints is unavoidable. If a line constraint is imposed on the flow through link 1–2, then (24) (in footnote 12) has to be satisfied. It is evident from (24) that, for this to happen, the sales by firm 2 at nodes one and two have to increase and those of firm 1 have to decrease. This explains the generation changes (in figure 1 from *square* to *diamond*) when a line constraint is imposed.

We can also observe the generation changes due to the imposition of an environmental constraint. In figure 1, the outputs "move" from *square* to *pentagram*. Generation by firm 1 diminishes substantially while that by firm 2 can increase (in period 2) or decrease a little (in period 1). In each period firm 2's market share is greater than in the unconstrained case.

We observe in figure 2 that imposing a constraint changes the proportions, in which the firms contribute to the total pollution. In this figure, each bar represents the total pollution corresponding to the base game, transmission-constraint game (*tc*), environmental-constraint game (*ec*) and transmission-&-environmental-constraint game (*tcec*), respectively. The two bottom fields symbolise the first period's pollution by firm 1 (bottom) and 2; the two top fields represent the second period's pollution by firm 2 (top) and 1.

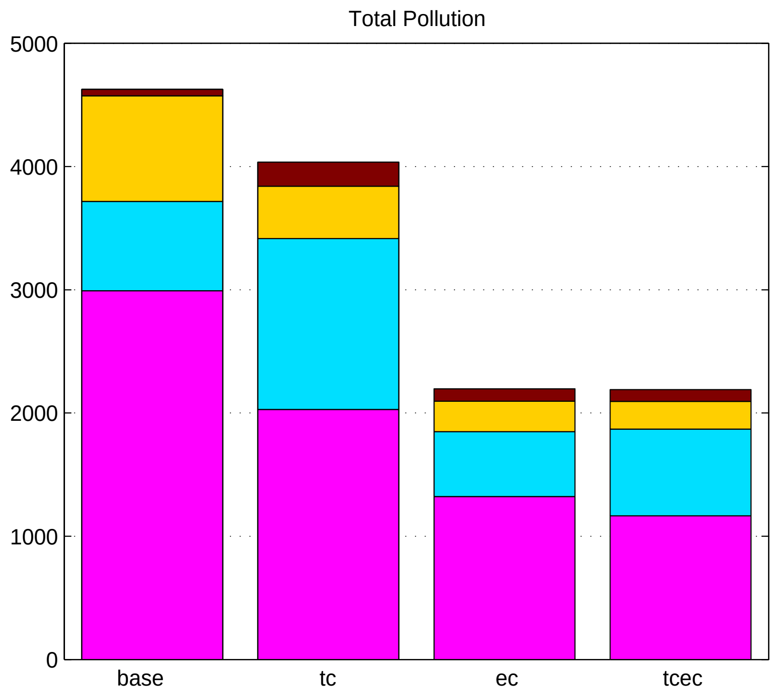

Figure 2: Total pollution in 1000lb.

The contributions are functions of the generation, see figure 3.

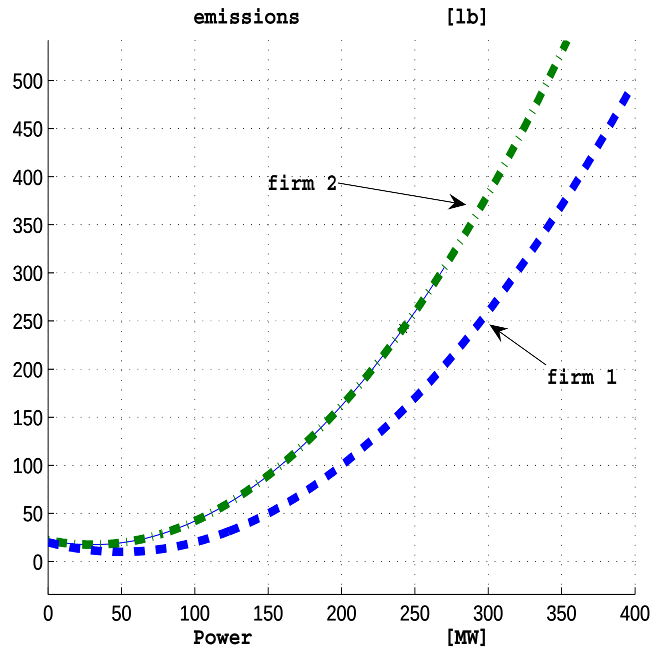

Figure 3: Firms' emissions as functions of hourly generations.

Given that firm 1 is more environmentally friendly (*i.e.*, it emits less per unit of generation) it could be expected that, as the environmental constraint tightens, firm 1's market share would increase. However, this is not what we observe in figure 1. As the environmental constraint is tightened, with or without a binding line constraint, the market share of the environmentally inefficient firm 2 increases. We conjecture that the source of this apparent anomaly are changes in the *effective* marginal costs $\mathbb{TMC}_f$ under the imposition of constraints. In words, the firm's effective marginal cost is $\mathbb{MC}_f$ $\Big($where $\mathbb{MC}_f$ is firm's f marginal cost $\dfrac{dC(P_f)}{dP_f}\Big)$ *plus* the marginal cost of the pollution constraints' violation. Under the constraint imposition we have[16]

$$\frac{\mathbb{MC}_1}{\mathbb{MC}_2} < \frac{\mathbb{TMC}_1}{\mathbb{TMC}_2} < 1 \ . \tag{31}$$

In broad terms, there is less difference between the firms when a constraint is binding hence the outputs' ratio "must" move closer to the 45^o line in figure 1.

Not surprisingly, if generation diminishes because of a constraint, the pollution diminishes too. Because of that, when contributions of firm 1 diminish, the relative contributions of firm 2 increase.

Overall, the result of the constraints' imposition is market distortion (see figure 1), more pollution coming form the less efficient firm (see figure 2) and higher energy prices (see the tables above). There are also significant imbalances between the weekdays and weekends pollution contributions.

[16]See [12].

The above analysis motivates the regulator to design and solve other generation games in which the *weights* r_f^t (see section 3.2, footnote 10) will modify players' efficient marginal costs so that the resulting equilibria will be "socially" more acceptable.

6 The second round's results

6.1 How to "improve" results?

The games' solutions will be obtained in this section for *various* weights $r_f^t > 0, f = 1, 2; t = 1, 2$ (*i.e.*, not all $r_f^t = 1$). The non identical weights will be used to help modify the players' efficient marginal costs in an attempt to achieve equilibria, socially more attractive than in section 5.

Accepting any set of weights r_f^t, including $r_f^t \equiv 1$, requires a political decision that is about the levels of responsibility for the common constraints' satisfaction. In particular, if all weights are set to 1, then the burden of the constraints' satisfaction is shared equally.

We have seen in section 5 that $r_f^t \equiv 1$ leads to equilibria that display several undesirable social characteristics. It is therefore conceivable that the regulator might be given a mandate (by the local parliament) to seek "better" equilibria, associated with an *unequal* treatment of the generators.

We will examine in this section whether it is possible for the regulator to diminish firm 2's market share gains and mitigate the emissions' asymmetry between the weekends and weekdays generations. We will also watch whether these solutions will result in lower prices for the consumers and, hence, higher consumers surplus than in section 5.

6.2 Generation under transmission and environmental constraints with preferences for pollution in the second period

An analysis of the results of section 5.5 concludes that due to a decreased economic activity in the second period, the pollution *stock* generated by the weekdays' output exceeds the weekend's stock more that five times. Consequently, the emission *flows* substantially differ between weekdays and weekends. While the overall pollution constraint (30) remains satisfied, the heavy flow on weekdays might create some health hazards. Also, the fact that it is rather uneconomical to sell electricity on weekends in the "rural" area can cause social problems.

We will examine whether the regulator can compute an equilibrium where there would be more production on weekends. We will set the weights r_1^2=100 and r_2^2=100 and keep $r_f^1 = 1,\ f = 1, 2$. The results obtained are shown in tables IX and X are visualized in figure 4.

Table IX: Generation and sales with a line flow limit of 25 MW in line 1–2 and an emission constraint with preferences for the second period

Sales by Firm 1 (1000MW)			Sales by Firm 2 (1000MW)			Generation by Firms (1000MW)		Emissions by Firms (1000lb)	
s_{11}^t	s_{12}^t	s_{13}^t	s_{21}^t	s_{22}^t	s_{23}^t	P_1^t	P_2^t	E_1^t	E_2^t
541.46	496.28	481.09	383.83	338.65	236.71	1518.8	959.2	992.35	585.12
233.47	190.01	198.93	156.11	112.65	78.99	622.4	347.8	420.19	192.33
774.94	686.29	680.02	539.95	451.31	315.71	2141.3	1307	1412.55	777.45

Table X: Economics of generation per hour per period with a line flow limit of 25 MW in line 1–2 and an emission constraint with preferences for the second period

Demand (MW)			Prices ($/MWh)			Profits ($/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
147.88	133.43	114.72	28.17	29.32	26.08	3127.85	1235.89
155.64	120.92	111.03	20.66	22.74	19.7	1489.81	155.41
Flows for lines (MW)			Lagrange multipliers		Emission levels (lb/h)		
$1-2$	$1-3$	$2-3$	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
25	69.86	44.86	16.26	-	158.6	93.51	252.11
25	68.02	43.02	1173.4	0.0041	167.87	76.84	244.71

The energy demanded per node for the first period is: 925.3 GWh, 834.9 GWh, and 717.8 GWh and for the second period, 389.6 GWh, 302.7 GWh, and 277.9 GWh. We notice that demand in the first period decreased and increased in period two.

The corresponding prices at the nodes are: 28.17, 29.32 and 26.08 $/MWh for the first period and 20.66, 22.74 and 19.7 $MWh for the second period. It is evident that the adopted weights helped ease the second period prices (compare the prices in table VIII).

The flows' absolute values per hour through the lines are: 25 MW, 69.9 MW, and 44.9 MW for lines 1–2, 1–3 and 2–3, in the first period and 25 MW, 68 MW, and 43 MW for lines 1–2, 1–3 and 2–3 in the second period. Profits for the first period for firms 1 and 2 are 19.57 M$ and 7.73 M$, respectively; for the second period, they are 3.72 M$ and 389 k$, respectively.

The Lagrange multipliers for the line and environmental constraints are: $\lambda_{L1}^1 = 16.26$; $\lambda_{L1}^2 = 1173.4$; $\lambda_E = 0.0041$.

Total pollution is kept at the constraint and equals 2,190.0 klb. As expected, the pollution in the second period has increased. The pollution changes between periods are presented in figure 4. In this figure as in figure 2, the bars represent the total pollution levels corresponding to the transmission-&-environmental-constraint game with $r_f^2 = 1$ (first bar) and $r_f^2 = 100$ (second bar). The two bottom fields together symbolise the first period's pollution by firm 1 (bottom) and 2; the two top fields together represent the second period's pollution by firm 2 (top) and 1. The pollution in period one has clearly diminished so has the contribution of firm 2 ("inefficient"). However, we also notice that this equilibrium is achieved with higher prices than before thus the consumers surplus is bound to diminish.

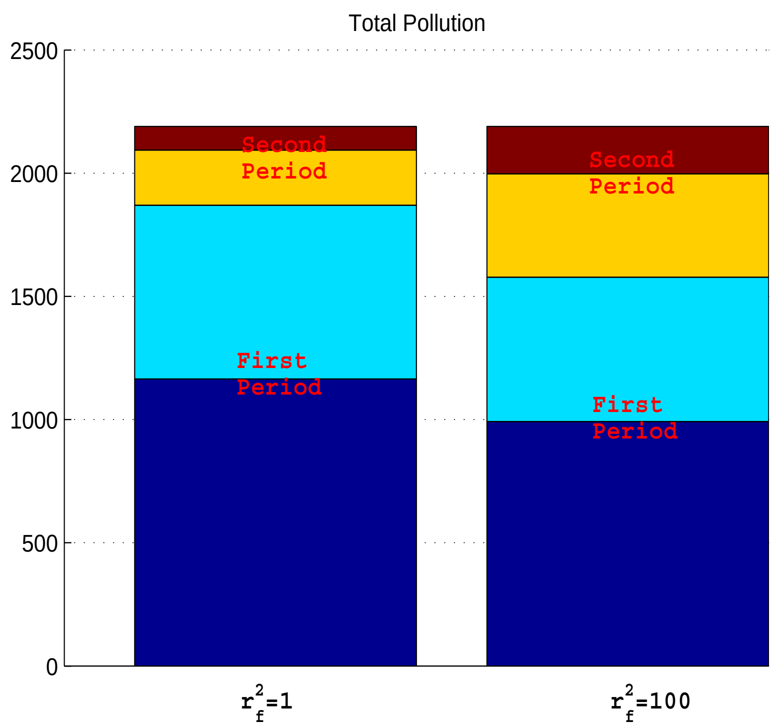

Figure 4: Total pollution in 1000lb as a function of the weights r^t.

We attribute the shift in generations between the first and second periods to the weights > 1 attached to payoffs in the second period. This means that, because of r_1^2=100 and r_2^2=100, the firms perceive the marginal cost of violating the environmental constraint in period 2 as $\frac{\lambda_E}{100}(\beta_f + 2\gamma_f P_f^2)$, rather than as $\lambda_E(\beta_f + 2\gamma_f P_f^1)$ in the first period.

6.3 Generation under transmission and environmental constraints with preference for the first firm

We will examine whether the regulator can compute an equilibrium where there would be more production assigned to player 1. We will set the weights $r_1^1 = r_1^2$=100 for player 1 and keep $r_2^1 = r_2^2$=1 for player 2. The results obtained are shown in tables XI and XII. We can observe that the profits have dramatically decreased for firm 2 in both periods.

Table XI: Generation and sales with with preference for the first firm

Sales by Firm 1 (1000MWh)			Sales by Firm 2 (1000MWh)			Generation by Firms (1000MWh)		Emissions by Firms (1000lb)	
s_{11}^t	s_{12}^t	s_{13}^t	s_{21}^t	s_{22}^t	s_{23}^t	$P_1^t\Delta^t$	$P_2^t\Delta^t$	E_1^t	E_2^t
395.8	698.9	674.2	801.4	0	0	1769	801.4	1418	410.5
142.6	218.2	166	207.2	0	0	526.8	207.3	282.8	78.7
538.4	917.1	840.2	1008.7	0	0	2295.8	1008.7	1700.8	489.2

Table XII: Economics of generation per hour per period with preference for the first firm

Demand (MW)			Prices ($/MWh)			Profits ($/h)	
σ_1^t	σ_2^t	σ_3^t	p_1^t	p_2^t	p_3^t	Π_1	Π_2
191.33	111.7	107.75	24.69	31.06	26.44	3640.24	601.08
139.8	87.17	66.32	21.61	24.77	21.43	1654.81	133.44
Flows for lines (MW)			Lagrange multipliers		Emission levels (lb/h)		
$1-2$	$1-3$	$2-3$	λ_T^t	λ_E	Firm$_1$	Firm$_2$	Total
25	66.37	41.37	2342.3	-	226.63	65.59	292.22
25	45.65	20.65	505.9	0.2	112.98	31.44	144.42

The energy demanded per node for the first period is: 1,197.2 GWh, 698.9 GWh, and 674.2 GWh and for the second period, 349.9 GWh, 218.2 GWh, and 166 GWh.

However, prices are hight. They are (at nodes 1, 2 and 3): 24.69, 31.06 and 26.44 $/MWh for the first period and 21.61, 24.77 and 21.43 $MWh for the second period. We will analyse the shifts in the consumers surplus for this and other cases in section 7.1.

The flows' absolute values per hour through the lines are: 25 MW, 66.37 MW, and 41.37 MW for lines 1–2, 1–3 and 2–3, in the first period and 25 MW, 45.65 MW, and 20.65 MW for lines 1–2, 1–3 and 2–3 in the second period.

Profits for the first period for firms 1 and 2 are 22.77 M$ and 3.76 M$, respectively; for the second period, they are 4.14 M$ and 334 k$, respectively (as opposed to M$ 19.74, 7.66, 3.74 and .733, respectively, for $r_f^t \equiv 1$, see section

5.5). The Lagrange multipliers for the line and environmental constraints are: $\lambda_{L1}^1 = 2342.3$; $\lambda_{L1}^2 = 505.9$; $\lambda_E = 0.2$.

Total pollution is equal to the upper bound, 2,190.0 klb.

The effect of incentivizing production of firm 1 is clearly observed in figure 5 where the pollution levels of firm 1 in both periods are substantially higher than in the case with all $r_f = 1$ (see the bottom bar and the second from the top for first and second period, respectively; notice that the bars do not start from zero).

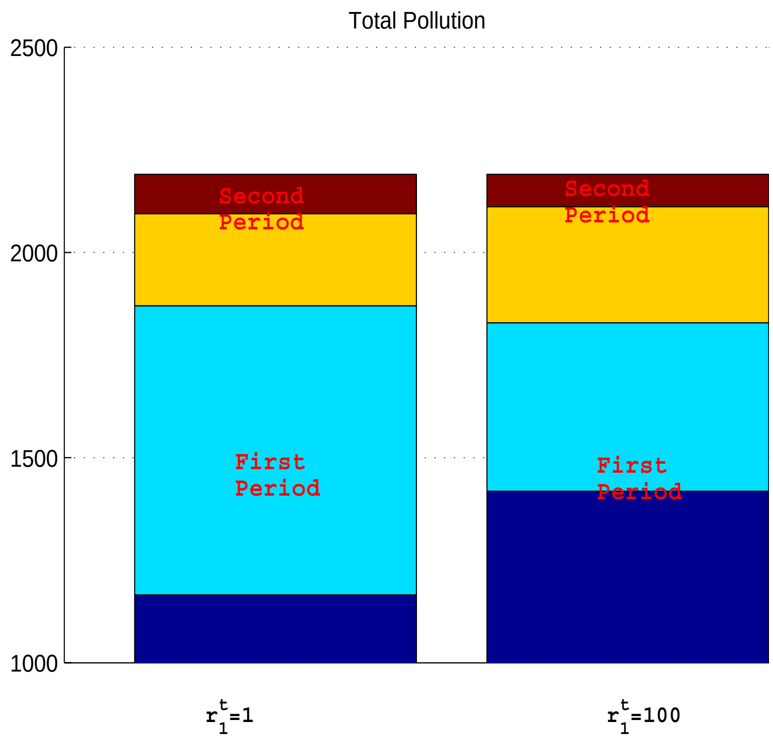

Figure 5: Total pollution in 1000lb as a function of the weights r^t.

7 What can the regulator learn from the above results

7.1 Implications for producer profits and consumer surplus

Figure 6 shows the plots of producer profits and consumer surplus (top and bottom respectively) as the constraints vary.[17]

[17]Producer profits (or producer surplus) are calculated from equation (1). Consumer surplus is $CS = \sum_{i=1}^{N} \frac{q_i^*}{2}(a_i - p_i^*)$ where q_i^* and p_i^* are the equilibrium quantity and price at node i and a_i is the price intercept of the demand curve.

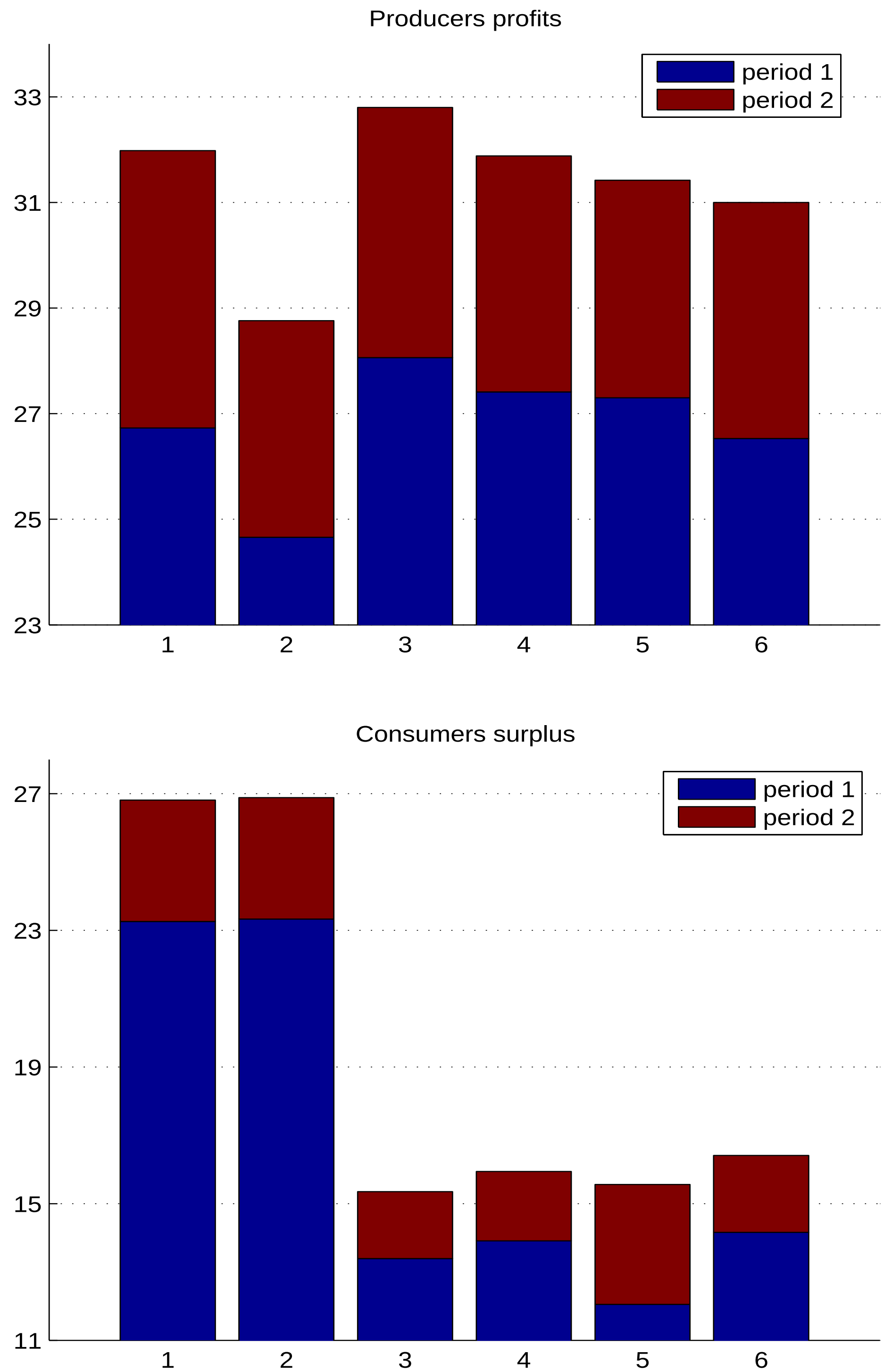

Figure 6: Producers' profits and consumers' surplus in M$.

There are six bars in each of the panels of figure 6 for the cases as follows:

1. the base case,
2. generation subject to transmission constraints,
3. generation subject to emission constraints,
4. generation subject to both constraints,
5. generation subject to both constraints with preferences for pollution in the second period,
6. generation subject to both constraints with preference for the first firm.

We observe that the effect of a transmission constraint is negligible in terms of consumers surplus because prices and overall demands remain fairly stable. However, it diminishes producers profits because of the 25 MW limit imposed, specially on the more efficient/cheaper firm 1.

The introduction of an emission constraint introduces higher prices. This causes consumers surplus to substantially decrease and producers profits to increase. If both constraints are imposed simultaneously the emission limit effect surpasses that of the line limit by far so, this case is almost equivalent to the emission-limit-alone case.

If production in period 2 is incentivized, prices increase in period 1 and decrease in period 2 making consumers surplus decrease in period 1 and increase in period 2. Producers profits remain almost the same since there is a small shift of generation between periods 1 and 2.

Finally, if production of firm 1 is incentivized the effects on both the producers profits and the consumers surplus appear clear but rather small. We notice the overall decrease in the producer profits and an increase in the consumers surplus. The absolute sizes of those changes relative to case 4 depend on the parameter values chosen for this case study. Hence, notwithstanding the small changes, the regulator might be pleased that the consumers surplus improves when firm 1 is incentivized.

7.2 Results of varying the "sharing rules"

The results obtained for the case study indicate that the regulator can impact the producer profits and consumer surplus by imposing different levels of the responsibility for the joint constraints' satisfaction. In other words, varying the rules for how the burden of the environmental and transmission standards' fulfilment is shared among the producers can generate a number of equilibria of uneven social desirability. The implication of this is that a desired equilibrium can be chosen.

8 Concluding remarks

We have proposed a methodology for the analysis of the impact of various constraints on electricity generation. In particular, this analysis should be useful for a regional government that is interested in an assessment of electricity supply changes due to an introduction of environmental constraints. For the case study considered in the paper, we notice the possibility of some market distortion when transmission constraints exist.

Introduction of an environmental constraint, which many businesses fear, may actually increase the business profits and make the consumers worse off economically.

We believe that thanks to our analysis, the regional government's choices will be informed by the tradeoffs between constraint satisfaction, economic activity and electricity supply.

In the next paper we plan to extend the analysis to include the optimal choice of environmental constraints.

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