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Infinite horizon hydroelectricity games

Jack Robles

SEF WORKING PAPER 11/2016

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Working Paper 11/2016

ISSN 2230-259X (Print)

ISSN 2230-2603 (Online)

Infinite Horizon Hydroelectricity Games*

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May 4, 2016

Abstract: We present an infinite horizon game between a hydro power plant and a thermal power plant. The Markov perfect equilibrium is characterized and related to the closed loop equilibrium of a one year finite horizon game. We show that the infinite horizon and one year horizon models make predictions which are surprisingly similar. Essential differences in predictions are noted.
JEL Classification Numbers: C73, D21, D43, L13
Keywords: Energy Market; Hydroelectricity; Markov Perfect Equilibrium; Cournot competition

*This paper has been presented at the Econometric Society Australasian Meetings and the IAEE conference. Thanks to conference participants, Joel Sobel and Vlado Petkov for helpful comments and conversations. Any remaining errors are my own.

1 Introduction

One of the defining features of electricity markets is the difficulty firms have in storing their product. As a consequence, a firm with convex costs or a capacity constraint cannot smooth production. Instead, firms may find themselves producing at high marginal costs in periods of high demand and low marginal costs in periods with low demand. Alternatively, lower cost firms may find themselves under-utilized when demand is low, and capacity constrained when demand is high. Hydroelectric power plants are the main exception. The reservoir of water behind a hydroelectric dam represents a store of energy which, subject to constraint, the hydro plant may release when most advantageous. Hydroelectric power plays a significant role in a number of electricity markets including: Brazil, California, Canada, Italy, Switzerland, and the Russian Federation.[1]

While we focus on unregulated markets, hydro plants play an important role in regulated markets as well. In a regulated market, a hydro plant can be used to supply peak demands allowing other plants to smooth their production. A hydro plant may fill this role to a greater or lesser extent in an unregulated market. On the other hand, absent regulation, a hydro plant might be used strategically to exasperate the variability in demand.

Because a hydro power generator's problem is, in every period, inherently dynamic, one needs an infinite horizon model to understand these markets. In this paper, we provide such a model, characterize its equilibria, and compare it to the (one year) finite horizon models which have been the workhorse for policy studies. Our model has two firms. Hydro owns a large hydro power producing dam. Thermal owns a plant that generates power by burning e.g. natural gas. The two firms compete in each period

[1] This list is far from exhaustive. Economies are chosen for having both: a large use of hydroelectric power, and a large GDP.

with quantity competition a la Cournot. Thermal behaves in a perfectly static manner, treating each period as a separate optimization problem. However, Hydro's problem is inherently dynamic. In any period, Hydro has a stock of water, and water not used in the current period can be saved for future use. We treat Hydro's stock of water as a state variable, and characterize a Markov Perfect Equilibrium.

Central to the analysis of Hydro's problem are a pair of constraints. Hydro uses her reservoir to pass water from wet low demand periods to dry high demand periods. She does this costlessly, but with constraint. She can not pass water backwards through time. This leads to a *current capacity constraint* which prevents her from using more water than is currently in her reservoir. She is also limited in her ability to pass water forward through time by the size of the reservoir. This leads to an *overflow constraint*, which requires that she generate sufficient power to prevent her reservoir from overflowing with excess water.[2]

As is typical, we assume that Hydro has no variable operating costs. Instead, Hydro acts to balance marginal revenue with the shadow value of water. This shadow value is the marginal future revenue forgone by using the water now. Absent the consequences of binding constraints, this leads Hydro to balance marginal revenues across periods. Consequently, hydro power output is typically larger in higher demand periods, which does allow some smoothing by thermal generators. However, this is still a far cry from the first best case in which Hydro acts to balance price across periods.[3] More generally, binding constraints bias output upwards in some periods. In general, the bias is greatest in the period in which the constraint binds. The bias becomes weaker as one moves backwards in time away from the next period with a binding constraint.

[2]If there are sufficient water inflows, then Hydro may wish to spill some of her water. On the other hand, spilling may be illegal. We do not allow spilling.

[3]Bushnell [3] finds a quite significant gap between the two cases in the California electricity market.

Realism requires that our model differ from the typical dynamic game. It is not typical for a player of a dynamic game to face the constraints that Hydro faces. In addition, we can not assume that every period is the same as the previous. Both water inflow to the reservoir and electricity demands vary significantly through a year. If one includes this variability, then a steady state is impossible. Instead we break time up into years. We allow that exogenous variables might differ through a year, but assume that each year replicates the last. We look for a 'steady cycle' in which behavior is the same from year to year.

Because a yearly cycle in exogenous variables is a somewhat unusual assumption, we take a moment to motivate it. A yearly cycle is a natural generalization of both dynamic games with constant exogenous parameters, and finite horizon models. Furthermore, since a model 'year' need not correspond to a calendar year, our model can accommodate multi-year cycles of drought and plenty. Finally, absent a yearly cycle in exogenous variables, we would not be able to solve for a steady cycle, which would make comparisons with the commonly used finite horizon models questionable.

We provide results characterizing steady cycles. We show that if a Markov Perfect Equilibrium converges to a steady cycle, then it must do so in finite time. We relate steady cycles to the equilibria of the (one year) finite horizon model developed in Crampes and Moreaux [4] and elaborated on by Robles [17] (CM-R henceforth). Steady cycles look in many ways like a subgame perfect equilibrium of an appropriately written year long finite horizon model. In fact, for a large range of parameter values, the two make identical predictions. Even when the two models make different predictions, there remain strong qualitative similarities between their predictions. When one considers how easy they are to use, these results provide strong support for the use of year long finite horizon models. However, this support is not unqualified. In particular, we demonstrate that there is a sense in which the equilibria of a finite horizon model over emphasizes the importance of current capacity constraints, and under emphasizes the importance of overflow constraints.

The difference in emphasis on the two types of constraints creates a quantitative difference in predictions. Current capacity constraints bind in dry periods, while overflow constraints bind in wet periods. Since, as discussed earlier, energy production is biased upwards when constraints bind, the finite horizon model and infinite horizon model can make quite different predictions regarding period by period energy production. However, since one year models are generally used to study qualitative issues, this may not be that troubling of a consideration.

Our result relating the equilibria in finite and infinite horizon models is especially useful because, starting with Scott and Read [18], and Crampes and Moreaux [4], many modelers have found finite horizon models so useful. Finite horizon models are used to study: the value of pump storage (Crampes and Moreaux [5],) the choice of reservoir size (Haddad [14],) investment in thermal technology (Genc and Thille [13],) merger policy (Skar and Sorgard [20],) and environmental externalities (Villemeur and Vinella [6].) Borenstein and Bushnell [2] and Bushnell [3] use calibrated finite horizon models to study the electricity markets in California and the western United States. Forsund [9] is a text which uses finite horizon models to study many situations. Hansen [15], Mathiesen, Skar and Sorgard [21], and Rangel [16] study how adding uncertainty to a finite horizon model changes the resource manager's decisions.

We provide a characterization of the policy variable which determines Hydro's output as a function of the reservoir level. The derivative of this policy variable appears in the Euler equations which determine outputs. It also appears in the first order condition for finite horizon models. We show that in any given period this derivative is determined in one of two manners. If a constraint binds in that period, then the derivative of the policy variable is equal to one. Otherwise, the derivative is determined via a closed form equation applied to the derivative of the policy variable in the following period.[4] As a consequence, solving for a steady cycle requires only that one identify the dates on which constraints bind, and then solve a system of linear equations. On the other hand, absent this characterization, even a three period finite horizon model cannot be solved analytically for a closed loop equilibrium.

Some other infinite horizon hydro power models have been suggested. Garcia, Reitzes and Stacchette [11] introduced an infinite horizon model of competition between hydro power producers.[5] However, in this model demand is perfectly inelastic in every period, water comes in discrete units, and there are no thermal power producers. Evans and Guthrie [7] provide an infinite horizon model with uncertainty. However, there is no seasonality in demand or water inflows, water inflows are either equal to zero or one, and the reservoir has capacity to hold one unit of water. Furthermore, only perfectly competitive markets are analyzed. Bobtcheff [1] and Genc and Thille [12] both provide infinite horizon discrete time models with uncertainty. Neither model has seasonality, both papers find only computational results, and both papers consider only perfectly competitive markets. Finally, Evans, Guthrie and Lu [8] generalizes the model in Evans and Guthrie [7]. However, again only a perfectly competitive market is analyzed, and the results are computational.[6]

The rest of the paper is organized as follows: The model and Euler equation are presented in Section 2. Results characterizing the steady cycle are presented in Section

[4]This closed form solution is Equation 19 in the Appendix.

[5]This model is elaborated on by Garcia, Campos-Nanez and Reitzes [10] and Skar [19].

[6]Evans, Guthrie and Lu [8] provide closed form solutions for the output in each period as a function of the shadow price of water. However, there is no analytical solution for the shadow price.

3. Section 4 compares the steady cycle with the subgame perfect equilibrium of a finite horizon model. Section 5 provides discussion. Proofs are restricted to the Appendix.

2 Model and Euler Equation

We model an infinite horizon game in which a hydro plant and a thermal plant engage in period by period quantity competition. In period $t = 1, 2, \ldots,$ Thermal generates q_t units of electricity, and Hydro generates h_t units of electricity. Hydro's marginal cost is zero, and Thermal's is $mc = c + z \cdot q_t$. In period t, the price for electricity is $P_t = a_t - b_t(q_t + h_t)$ with $c < \min_t\{a_t\}$. We treat the quantity of water in the reservoir, R_t, as the state variable of our problem, and look for a Markov Perfect Equilibrium. Let ψ_t and U_t denote Thermal's instantaneous profits and value function for time t. Let r denote the discount rate. Thermal's objective is

$$\max_{q_t} \psi_t(h_t, q_t) + rU_{t+1}(R_{t+1}) \tag{1}$$

Because Thermal has no control of the state variable, his incentives are captured with the first order condition in q_t;

$$\frac{\partial \psi_t}{\partial q_t} = 0 \tag{2}$$

Since ψ_t is not a function of R_t, Thermal's decisions are static in nature. The choice of q_t is dynamic only through the first order condition's dependence on h_t. Let $Q_t(R_t, h_t)$ denote the dependence of Thermal's choice as a function of the state and Hydro's output. It follows that $\frac{\partial Q_t}{\partial R_t} = 0$.

Hydro has no marginal costs, but faces a resource constraint on water. In particular, Hydro's reserve of water evolves according to $R_{t+1} = R_t + w_t - h_t$, where w_t is the period t inflow of water. In period t, Hydro faces two constraints. The *current capacity constraint* requires that she not use more water than currently available, which requires $R_{t+1} \geq 0$ (multiplier κ_t.) Let $\bar{R}$ denote the maximum capacity of Hydro's reservoir. The *overflow constraint* requires that Hydro not allow her reservoir to overflow, which requires that $R_{t+1} \leq \bar{R}$ (multiplier θ_t.) We include neither turbine capacity constraints nor non-negativity constraints.[7] We argue in Section 5 that the inclusion of these constraints complicates the analysis and the statement of results without a compensating improvement in our understanding of behavior.

Let π_t and V_t denote Hydro's instantaneous profits and value function for time t. Hydro's objective is

$$\max_{h_t} \pi_t(h_t, q_t) + rV_{t+1}(R_{t+1}) \tag{3}$$

subject to: $R_{t+1} = R_t + w_t - h_t$, and $0 \leq R_{t+1} \leq \bar{R}$. Let $H_t(R_t, q_t)$ denote the solution to Hydro's problem. Fairly standard arguments lead us to the following.

Proposition 1 *If $\frac{dH_{t+1}}{dR_{t+1}}$ is well defined, then Hydro's Euler equation is*

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{dH_{t+1}}{dR_{t+1}}\right). \tag{4}$$

Before discussing Equation 4, we observe that $\frac{dH_{t+1}}{dR_{t+1}}$ is not always well defined. In particular, if a constraint is weakly binding in period $t+1$, then H_{t+1} is kinked. In this case we must work with left-hand and right hand-derivatives. Except when they prove illuminating, the details of this more general analysis are restricted to the Appendix.

Equation 4 has a straightforward interpretation. If we set all the multipliers to zero, the left-hand side is period t marginal revenue, and the right-hand side is the period t shadow cost. The shadow cost has two components. The first term is the (discounted)

[7]It has been suggested by others that the lack of a non-negativity constraint for Hydro might be taken as an assumption of a perfectly efficient pump storage system. We do not push this interpretation.

period $t+1$ marginal revenue. The second term is the (discounted) period $t+1$ strategic effect. Water unused in period t increases the period $t+1$ stock of water. This causes an anticipated increase in Hydro's output in period $t+1$, which decreases Thermal's output.

If one of the period t constraints binds, then behavior in period t is nailed down exactly by the need to satisfy that constraint. If the current capacity constraint binds, then all available water is used and $h_t = R_t + w_t$. In this case, $\frac{\partial \pi_t}{\partial h_t}$ is greater than the shadow price, and $\kappa_t \geq 0$ measures this disparity. If the overflow constraint binds, then just sufficient water to avoid an overflow is used and $h_t = R_t + w_t - \bar{R}$. In this case, $\frac{\partial \pi_t}{\partial h_t}$ is less than the shadow price, and $\theta_t \geq 0$ measures the disparity.

If, on the other hand, no period t constraints binds, then $\kappa_t = \theta_t = 0$, and the Euler Equation 4 becomes

$$\frac{\partial \pi_t}{\partial h_t} = r \left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{dH_{t+1}}{dR_{t+1}} \right) \tag{5}$$

If $\frac{dH_{t+1}}{dR_{t+1}} = 0$ then Equation 5 looks very much like the open loop solution in CM.[8] If $\frac{dH_{t+1}}{dR_{t+1}} = 1$, then Equation 5 looks very much like CM-R's closed loop solution when a constraint binds in the last period.

With this in mind, let us consider the possibility that a constraint binds in period $t+1$. If the $t+1$ overflow constraint binds, then any additional water passed onto period $t+1$ must be used immediately. If the $t+1$ current capacity constraint binds, then any small amount of water passed onto period $t+1$ will be used immediately because it has higher value in period $t+1$ than in later periods. In both cases $\frac{dH_{t+1}}{dR_{t+1}} = 1$. That is, if a constraint binds in period $t+1$, then period $t+1$ looks like the last period of

[8] Depending on parameter values, a closed loop equilibrium can have a similar solution.

a closed loop equilibrium of a finite horizon game. This similarity is the starting point for comparisons between the equilibria of finite and infinite horizon models.

The value of $\frac{dH_t}{dR_t}$ is not so obvious when a constraint does not bind in period t. Let us say that starting from period t, the first period in which a constraint binds is period $t+k$. It is trivially true that $0 \leq \frac{dH_t}{dR_t} < 1$, because otherwise an exogenous increase in R_t would upset the Euler equation. A more in depth analysis leads to the following.

Proposition 2 *Let t denote a period in which no constraint binds, and assume that $\frac{dH_{t+1}}{dR_{t+1}}$ is well defined.*
(A) If $\frac{dH_{t+1}}{dR_{t+1}} = 0$, then $\frac{dH_t}{dR_t} = 0$. If $\frac{dH_{t+1}}{dR_{t+1}} = 1$, then $\frac{dH_t}{dR_t} < 1$.
(B) $\frac{dH_t}{dR_t}$ is a strictly increasing function of $\frac{dH_{t+1}}{dR_{t+1}}$ on the range $[0, 1]$.
(C) $\frac{dH_t}{dR_t}$ does not depend directly upon R_t.

The importance of Statements (A) and (B) can be seen in the the fact that the strategic effect in the period t Euler Equation 4 is increasing in $\frac{dH_{t+1}}{dR_{t+1}}$. Hence, this effect is strongest when a constraint is binding in period $t+1$. Furthermore, the strength of the strategic effect tends to decrease the further in the future one must go to find a binding constraint. Finally, Statement (A) is used in the sequel to demonstrate that if no constraint binds at any time in the future, then the strategic effect disappears entirely. Statements (A) and (B) should hold more generally, because otherwise an exogenous increase in R_t would upset the period t Euler equation. On the other hand, statement (C) is special to the linear model, and may be a bit misleading. Obviously $\frac{dH_t}{dR_t}$ must depend upon R_t. However, in a linear model $\frac{dH_t}{dR_t}$ depends upon R_t only indirectly, through the role that R_t plays in determining the next period in which a constraint binds.

2.1 Alternative Market Structures

To give an indication of how our results would change under different market structures, we briefly consider two alternatives. This may also serve to illuminate features of our current Euler Equation 4.

We consider first the possibility that Hydro is a monopoly in energy production. Because a firm that sets quantity ala Cournot acts like a monopolist on its residual demand, this resulting Euler maintains some similarity to Euler Equation 4.

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t = r \left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} \right). \tag{6}$$

Of course, the marginal revenue term, $\frac{\partial \pi_t}{\partial h_t}$, in Equation 6 is taken on total period t demand rather than the residual period t demand after Thermal's output has been counted. The obvious difference between Euler Equations 4 and 6 is the absence of the strategic effect in Equation 6. This term disappears, because there is no Thermal plant to influence.

A more dramatic alteration of the model is to imagine that Hydro is a price taker. A price taking firm treats the price it faces as its marginal revenue. In addition, a price taking firm is not aware of the impact its output has on its rivals. Hence, if Hydro is a price taker, its Euler Equation would be

$$P_t - \kappa_t + \theta_t = rP_{t+1} \tag{7}$$

Because a price taker ignores strategic effects, this Euler Equation holds irrespective of the structure of the rest of the market. We conclude that a price taking Hydro acts to maximize welfare by balancing the (discounted) value of the marginal unit of energy across periods.

It is important to note that in the two alternative cases that we consider the strategic term disappears. This removes the derivative of the period $t+1$ policy variable, $\frac{dH_{t+1}}{dR_{t+1}}$, from the period t Euler. Since much of the Appendix is spent dealing with this derivative, it stands to reason that the equilibria of these alternative models would be better behaved and easier to characterize and solve.

Given the above, it seems reasonable to compare the equilibria of the three models based upon the Euler Equations alone. We take the competitive model as the baseline because it leads to the social optimum. Going from the competitive to the monopoly case creates a bias in which output is lower in periods with less elastic demand.

Going from the monopoly case to the model in this paper adds another bias. Consider an interval of periods τ, $\tau+1$, ... $\tau+k$. Say that a constraint binds in period $\tau+k$, but in no earlier periods of the interval. All other things equal, adding the strategic term will tend to make outputs in later periods of this interval higher, and outputs in earlier periods of this interval lower.

3 The Steady Cycle

As stated in the introduction, we assume a 'yearly' pattern in the exogenous variables. Within each 'year,' we allow $D \geq 1$ heterogeneous periods. Hence, in our analysis period t has the same exogenous details as period $t+D$. Let $d(t)$ denote the date (month and day) of period t. The date function $d(t)$ maps the strictly positive integers into the set $\{1, 2, \ldots D\}$. That is, if x is an integer such that $0 < j = t - xD \leq D$, then $d(t) = j$. A yearly pattern means that $P_t(\cdot) = P_{d(t)}(\cdot)$ and $w_t = w_{d(t)}$. Given a yearly cycle, we make one further characterization of $\frac{dH_t}{dR_t}$.

Proposition 3 *Let t denote a period in which no constraint binds, and τ the first period following t in which a constraint is strictly binding. Assume that no constraint binds weakly between t and τ.*
If there is a yearly pattern in exogenous variables, then $\lim_{\tau \to \infty} \frac{dH_t}{dR_t} = 0$.

The message to take from Propositions 2 and 3 is that the value of $\frac{dH_t}{dR_t}$ is determined entirely by which of the present and future constraints bind. Furthermore, the farther in the future one must go to find the next binding constraint, the closer $\frac{dH_t}{dR_t}$ is to zero. In the limit, when no current or future constraint binds, H_t is independent of R_t.

Example 1. We illustrate the behavior of $\frac{dH_t}{dR_t}$. Recall that period t demand is $P_t = a_t - b_t(h_t + q_t)$ and that Thermal's marginal costs are $mc = c + z \cdot q_t$. It is shown in the Appendix that the relationship between $\frac{dH_t}{dR_t}$ and $\frac{dH_{t+1}}{dR_{t+1}}$ is determined by z, b_t, b_{t+1}, and r. For Table 1, we set $z = 0$ and $r = 0.9$, and presume that a constraint binds in period $t = 8$.

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$
b_t	1	2	1	2	1	2	1	2
$\frac{dH_t}{dR_t}$	0.09	0.05	0.13	0.08	0.22	0.17	0.55	1

Table 1: Backwards Decay of $\frac{dH_t}{dR_t}$

We note that $\frac{dH_t}{dR_t}$ is not monotonic in time. In particular, if $b_t < b_{t+1}$, then it is possible that $\frac{dH_t}{dR_t} > \frac{dH_{t+1}}{dR_{t+1}}$. On the other hand, if $b_t \geq b_{t+k}$ and no constraints bind between t and $t+k$, then $\frac{dH_t}{dR_t} < \frac{dH_{t+k}}{dR_{t+k}}$. This is confirmed in the Table 1 where $\frac{dH_t}{dR_t} < \frac{dH_{t+2}}{dR_{t+2}}$.

We look for a *Steady Cycle* of a Markov Perfect Equilibrium (MPE). A steady cycle is a yearly pattern that, once begun, a MPE might repeat indefinitely. The nature of a steady cycle depends on which constraints bind, and whether those constraints bind weakly or strictly. An *open cycle* is one in which neither the overflow nor the current capacity constraint ever binds. An *overflow cycle* (resp. *current capacity cycle*) is one in which the overflow constraint (resp. current capacity constraint) binds *strictly* on at least one date during every cycle. If no constraint binds strictly, but the current capacity constraint (resp. overflow constraint) binds weakly on at least one date, then we term this a weak current capacity cycle (resp. weak overflow cycle.) A cycle in which both current capacity and overflow bind is called a mixed cycle. On the other hand, if a steady cycle is, for example, a non-mixed current capacity cycle, then we call it a pure current capacity cycle.

We now consider convergence. If a MPE converges to a steady cycle, then it converges in finite time. Let $\hat{R}_d$ denote the reservoir level on date d once the steady cycle has been entered. Let R_t^* denote the reservoir level on period t in the MPE. We say that the MPE converges to the steady cycle if for each $\epsilon > 0$, $\exists T$ such that $|R_t^* - \hat{R}_{d(t)}| < \epsilon$ for $t > T$. The MPE converges in finite time if $\exists T$ such that $R_t^* = \hat{R}_{d(t)}$ for $t > T$.

Proposition 4 *If a MPE converges to a steady cycle, then it converges in finite time.*

This result is in contrast to typical results in which a MPE approaches, but never reaches, a steady state. It obtains because of the important role played by constraints. In particular, say that within the converged to steady cycle a constraint binds on a date d. The constraint must eventually bind along the MPE path, and once it does, the steady cycle has been entered.

We now characterize the various types of steady cycles. We begin with open cycles because they are the simplest to describe. Let $\mathcal{C}_t$ denote the value of h_t which sets $\frac{\partial \pi_t}{\partial h_t} = 0$ when $q_t = Q_t(\mathcal{C}_t)$. That is, $\mathcal{C}_t$ would be the Cournot output if the firms were playing a static game in period t in which Hydro faced no constraint. Once an open cycle has been entered, no constraints bind in the future which removes the strategic effect from the Euler equation. This leads to a rather static outcome, in which Hydro produces $\mathcal{C}_t$ in every period.

Proposition 5 *In an open cycle $h_d = \mathcal{C}_d$ on every date d.*

It is trivial to solve for an open cycle. On the other hand, an open cycle is possible only if $\overline{\mathcal{C}} \equiv \sum_{d=1}^{D} \mathcal{C}_d = \sum_{d=1}^{D} w_d \equiv \overline{W}$. Such a restriction fails any reasonable notion of genericity.

One can solve for non-open steady cycles by first identifying the dates on which constraints bind, and then solving the system of linear Euler equations. This task can be made more difficult if some constraints are weakly binding. A weakly binding constraint can create a kink in the (period $t+1$) strategy variable, which makes it more difficult to work with the (period t) Euler equation.

Kinks in the strategy variables do not create difficulties in pure (weak) overflow cycles. Of course, a weakly binding overflow constraint *can* create a kink. However, this kink creates a non-convexity; the point where an overflow binds weakly is never a local maximum. Hence if Hydro is free to change outputs so that the overflow is not weakly binding, then she will. In other words, if overflow is weakly binding in period $t+1$, then some constraint must be strictly binding in period t. This means that we can find the period t output by the need to satisfy the constraint, rather than by looking at the period t Euler equation. Hence, the kink in H_{t+1} has no impact on any Euler Equations used to calculate outputs. These arguments also imply that there are no pure weak overflow cycles.

Proposition 6 *There are no pure weak overflow cycles.*

On each date d of a pure overflow cycle, either the overflow constraint is binding or $\frac{dH_{d+1}}{dR_{d+1}}$ *is defined.*

Next we consider pure current capacity cycles. These are only possible when yearly inflows of water are relatively low. From a purely static point of view, Hydro wants to produce $\mathcal{C}_t$. The Euler Equation makes her want to produce less than she would in a static situation since there might be a strategic benefit from passing water to the future. Finally, the current capacity constraint puts no lower limit on her output. Hence, there is nothing to push period t output above $\mathcal{C}_t$.

Proposition 7 *In a pure current capacity cycle,* $h_d < \mathcal{C}_d$ *on every date* d*.*

There is some sense in which Proposition 7 is weaker than Propositions 6. We are unable to rule out weak current capacity cycles. What is more, a weakly binding current capacity constraint within a current capacity cycle need not be isolated by being preceded by a strictly binding constraint. On the other hand, Proposition 7 implies an upper bound on inflows within a current capacity cycle. An example below demonstrates that this upper bound can be quite low.

Weak current capacity cycles look either like open cycles or like pure current capacity cycles.

Proposition 8 *In a weak current capacity cycle either:*
(A) $h_d = \mathcal{C}_d$ *on every date of the cycle, or*
(B) $h_d < \mathcal{C}_d$ *on every date of the cycle.*

Case (A) of Proposition 8 is essentially an open cycle in which the reservoir level drops to zero on some date of the cycle. Clearly the weakly binding constraint on that date has no impact on behavior. Case (B) replicates the result for current capacity cycles. CM-R show that in some cases, finite horizon models have a subgame perfect equilibrium in which $\mathcal{C}_t$ is produced in the last period, but less than $\mathcal{C}_t$ is produced in preceding periods. By Proposition 8, MPE can not replicate this sort of behavior.

We turn now to the question of how the total yearly inflows determine the steady cycle. Propositions 5, 7 and 8 immediately imply the following.

Proposition 9 *If $\overline{W} > \overline{\mathcal{C}}$, then a steady cycle must be an (possibly mixed) overflow cycle.*

The case in which $\overline{W} \leq \overline{\mathcal{C}}$ is more complicated. In general, when $\overline{W} \leq \overline{\mathcal{C}}$ we might have multiple steady cycles, a unique steady cycle, or no steady cycles.[9] The cycles, when they exists, may be either overflow cycles or current capacity cycles.

Conditions under which pure (weak) current capacity cycles exist can be restrictive.

Proposition 10 *Fix details of the model other than the discount rate. There exists some $\overline{r} < 1$ such that if $r > \overline{r}$ then every steady cycle has a date on which an overflow constraint binds.*

In general, the larger are yearly inflows, the easier it is to satisfy the conditions for an overflow cycle, and the harder it is to satisfy the conditions for a current capacity cycle, or a weak current capacity cycle. Proposition 10 tells us that as r gets close to 1 it becomes harder to satisfy the requirements for a current capacity cycle. The following example illustrates.

Example 2 We set Thermal's marginal cost to $mc^T = 1 + q/10$, and demand parameters to $b_1 = b_2 = 1$, $a_1 = 4.5$, and $a_2 = 5$. All inflows occur in odd periods, i.e. $w_2 = 0$.

[9] We do not know if this implies non-existence of MPE. On the one hand, there may be a MPE which does not converge to a steady cycle. On the other hand, because of Hydro's constraints, we do not have linear quadratic payoffs. Consequently, we have no guarantee of existence for MPE.

Hence, overflow cycles have a constraint binding in odd periods, and current capacity cycles have a constraint binding in even periods. For a given value of r, there is a lower bound $\overline{W}^{of}$ such that there is an overflow cycle if $w_1 > \overline{W}^{of}$. Likewise there is a $\overline{W}^{cc}$ such that there is a current capacity cycle if $w_1 < \overline{W}^{cc}$. Finally there is a $\overline{W}^{wcc}$ such that there is a weak current capacity cycle if $w_1 < \overline{W}^{wcc}$. We ignore the cycles which are possible only if $\overline{W} = \overline{\mathcal{C}}$. Table 2 illustrates these bounds for different values of r. In this example $\overline{\mathcal{C}} = 3 + \frac{5}{6}$. The first thing to note on the table is that

r	0.90	0.92	0.94	0.96	0.98
$\overline{W}^{\mathrm{of}}$	1.358	1.156	0.931	0.680	0.399
$\overline{W}^{\mathrm{cc}}$	1.246	1.037	0.804	0.544	0.251
$\overline{W}^{\mathrm{wcc}}$	1.020	0.847	0.660	0.458	0.239

Table 2: Inflow Boundaries

$\overline{W}^{wcc} < \overline{W}^{cc} < \overline{W}^{of} < \overline{\mathcal{C}}$ in each case considered. Hence, there is a range over which a steady cycle does not exist.[10] In addition, there is a large range over which there is a weak current capacity cycle; weakly binding constraints cannot be ruled out with a genericity assumption. On the other hand, whenever there is a weak current capacity cycle, there is also a current capacity cycle. Within this example at least, one need never feel obliged to create a steady cycle with any weakly binding constraints. Finally, as indicated by Proposition 10, as $r \to 1$ current capacity cycles become possible only when water becomes progressively scarcer and scarcer.

[10]There is a range of inflow over which existence fails so long as $a_1 < a_2$. If $a_1 > a_2$, then there would be a range within which there were multiple steady states.

4 Comparison to Finite Horizon Models

In this Section we present two results relating steady cycles to subgame perfect equilibria of finite horizon models. The first result starts with a MPE which converges to a steady cycle, and then states necessary condition under which a finite horizon model might have an equilibrium which mimics this steady cycle. The second result starts with an equilibrium of a finite horizon model and works in the opposite direction. We work with necessary conditions to rule out implausible equilibria.

We introduce some notation. Denote a period within the finite horizon model by γ. A superscript F indicates that something is an element of the finite horizon model. For example, $P^F_\gamma(\cdot)$ is demand in period γ of the finite horizon model. We say that the finite horizon model is a (year long) run (of the infinite horizon model) ending on date d if (1) there are D periods in the finite horizon model, and (2) $\{P^F_\gamma(\cdot), w^F_\gamma\} = \{P_{d+\gamma}(\cdot), w_{d+\gamma}\}$. Let (h^F_γ, R^F_γ) (resp. (h_t, R_t)) denote equilibrium values for the finite (resp. infinite) horizon model. We say that the equilibria of the finite and infinite horizon model mimic each other if there is a τ such that (1) the finite horizon model is a run ending on date $d(\tau)$, (2) the equilibrium of the infinite horizon model has entered a steady cycle by date τ, and (3) $\{h^F_\gamma, R^F_\gamma\} = \{h_{\tau+\gamma}, R_{\tau+\gamma}\}$ for $1 \leq \gamma \leq D$.

Comparisons between the finite horizon and infinite horizon cases are facilitated by a *large reservoir assumption.* Let $\bar{P}_t = P_t(Q_t(0, R_t))$ denote the period t price if Hydro produces nothing and Thermal chooses a best response. Let
$\rho_t = \max\{(\bar{P}_{t-1} - r \cdot \bar{P}_t)/(r \cdot b_t), 0\}$.

Assumption 1 *The reservoir is sufficiently large;* $\overline{R} > \max_t\{\max\{w_t + \rho_t, 2\mathcal{C}_t\}\}$.

Assumption 1 does not rule out mixed cycles. However, it does rule out MPE in which the reservoir switches rapidly back and forth between empty and full. The restriction that $\overline{R} > w_t + \rho_t$ assures that the reservoir can't go from empty to full in one period. Likewise, $\overline{R} > 2\mathcal{C}_t$ assures that it takes more than two periods of moderate water use to empty a full reservoir.

We consider an equilibrium of a finite horizon model and a steady cycle which mimic each other. For simplicity of discussion, let us say that the finite horizon model is a run ending on date D and that we are considering a MPE which has converged by period 1. We observe that we can think of a finite horizon model as an infinite horizon model in which V_γ is constant for $\gamma > D$. This implies that for each period $t \in \{1, ..., D-1\}$ Euler equation of the infinite horizon model, there is an equivalent $\gamma = t$ first order condition in the finite horizon model. However, the period D first order condition is in general different from the period D Euler equation. In a steady cycle, period $D+1$ and period 1 are identical. Hence, the following relationship between period 1 and period D holds:

$$\frac{\partial \pi_D}{\partial h_D} \left\{ \begin{array}{c} = \\ > \\ < \end{array} \right\} r \left(\frac{\partial \pi_1}{\partial h_1} + \frac{\partial \pi_1}{\partial q_1} \cdot \frac{\partial Q_1}{\partial h_1} \cdot \frac{\partial H_1}{\partial R_1} \right) \text{ if } \left\{ \begin{array}{c} \text{no constraint binds strictly} \\ \text{current capacity binds strictly} \\ \text{overflow binds strictly} \end{array} \right\} \tag{8}$$

On the other hand, in the finite horizon equilibrium, water left over from period D has no value. This leads to

$$\frac{\partial \pi_D^F}{\partial h_D^F} \left\{ \begin{array}{c} = \\ > \\ < \end{array} \right\} 0 \text{ if } \left\{ \begin{array}{c} \text{no constraint binds strictly} \\ \text{current capacity binds strictly} \\ \text{overflow binds strictly} \end{array} \right\} \tag{9}$$

A second issue is that in a steady cycle of the infinite horizon model $R_1 = R_{D+1} =$

$R_D + w_D - h_D$. On the other hand, in the finite horizon model there is no period $D+1$, and no equilibrium relationship imposed on R_1.

If a steady cycle can be mimicked, then we say that steady cycle is *representable.* The necessary conditions under which a steady cycle is representable are made overly complicated by the presence of a number of knife's edge cases. We rule these out with a *genericity* assumption. In particular, we would like to focus on equilibria which are not excessively sensitive to the size and timing of inflows to the reservoir. With that in mind, we perturb $\{w_d\}_{d=1}^D$, and hold other parameters of a model constant. We say that $\{\tilde{w}_d\}_{d=1}^D$ is an ϵ-perturbation of $\{w_d\}_{d=1}^D$ if $\sum_{d=1}^D |w_d - \tilde{w}_d| < \epsilon$. Let us denote a particular steady cycle given inflows $\{w_d\}_{d=1}^D$ as E and a particular steady cycle given inflows $\{\tilde{w}_d\}_{d=1}^D$ as $\tilde{E}$. We say that the steady cycles E and $\tilde{E}$ are congruent if a particular constraint (current capacity or overflow) binds in a particular manner (weakly or strictly) on date d in steady cycle E if and only if it binds in that same manner on the same date in steady cycle $\tilde{E}$. A steady cycle is *generically representable* if there exists an $\epsilon > 0$ such that every ϵ-perturbation has a congruent steady cycle which is also representable.

Proposition 11 *Let the Large Reservoir Assumption hold. Consider a MPE which converges to a steady cycle prior to period $T - D$. This steady cycle is generically representable only if one of the following holds.*
(A) $\exists \tau \geq T$ such that the overflow constraint binds strictly in period τ, and $h_\tau > \mathcal{C}_\tau$.
(B) $\exists \tau \geq T$ such that current capacity binds strictly in period τ, and $h_\tau < \mathcal{C}_\tau$.

Proposition 11 does not allow for weak current capacity cycles. Weak current capacity cycles *can* be generic to the perturbations we consider. However, weak current capacity cycles are not representable. As we will see in Example 3 below, equilibria with only a weakly binding current capacity constraint look very different in a finite horizon and infinite horizon model.

It is worth mentioning that Proposition 11 Case (B) is automatically satisfied by a MPE which converges to a pure current capacity cycle. On the other hand, Case (A) is satisfied by a MPE converging to a pure overflow cycle only if yearly inflow are sufficiently large.

We say that a subgame perfect equilibrium which can be mimicked by a steady cycle is valid. There is one immediate observation we can make about valid equilibria. Since inflows of water must equal outflows of water over the course of a steady cycle, the same equivalence must hold for a valid subgame perfect equilibrium. This gives us our first necessary condition.

Proposition 12 *A subgame perfect equilibrium of a finite horizon model is valid only if* $R_1^F = R_D^F + w_D^F - h_D^F$.

Because Hydro only pays attention to the sum $R_1^F + w_1^F$, Proposition 12 is fairly weak.

Valid equilibria include knife's edge cases that we wish to rule out. We follow an approach parallel to that which ruled out knife's edge cases for representable steady cycles. The notions of an ϵ perturbation and a congruent equilibrium can be applied to the finite horizon model without modification. Again an ϵ perturbation occurs if inflow are changed so that the sum of absolute differences in period by period inflows is less than ϵ. Again, we say that a pair of subgame perfect equilibria are congruent if constraints bind in the same periods, and in the same manner. A valid equilibrium is *generically valid* if every sufficiently small perturbation has a congruent equilibrium which is also valid.

It is immediate that a generically valid subgame perfect equilibrium is mimicked by a generically representable steady cycle. This leads to a second rather immediate observation: a generically valid equilibrium must have a period on which overflow (resp. current capacity) binds strictly, and on which $h^F_\gamma > \mathcal{C}^F_\gamma$ (resp. $h^F_\gamma < \mathcal{C}^F_\gamma$.) We can always arrange the start date of the year so that one of these conditions occurs on the final period. However, we can make a stronger statement.

Proposition 13 *Let the Large Reservoir Assumption hold. Consider a subgame perfect equilibrium of a finite horizon model in which $\frac{dH^F_1}{dR^F_1}$ is defined. The equilibrium is generically valid only if one of the following conditions holds*
(A) Overflow binds strictly in D^F, and Eqn. 8 holds with a '<.'
(B) Current capacity strictly binds in period D^F, and Eqn. 8 holds with a '>.'

Recall that Equation 8 holds with a '<' (resp. '>') when overflow (resp. current capacity) is strictly binding on date D of the mimicking steady cycle.

In contrast to Proposition 11, neither Case (A) nor Case (B) is automatically satisfied when the appropriate constraint binds in period D^F. However, in the typical finite horizon models constraints bind only in the final period. If overflow binds strictly in the final period, and no other constraint binds, then Case (A) is satisfied. In this event, Euler Equation 5 applies in each period $\gamma < D^F$. Consequently, for $\gamma < D^F$,

$$\frac{\partial \pi^F_\gamma}{\partial h^F_\gamma} \geq \min\left\{\frac{\partial \pi^F_{\gamma+1}}{\partial h^F_{\gamma+1}}, 0\right\}.$$

Since the overflow binds strictly in period D^F, it follows that $\frac{\partial \pi^F_D}{\partial h^F_D} < 0$. As a result, $\frac{\partial \pi^F_D}{\partial h^F_D} < r\frac{\partial \pi^F_1}{\partial h^F_1} < r\left(\frac{\partial \pi_1}{\partial h_1} + \frac{\partial \pi_1}{\partial q_1} \cdot \frac{\partial Q_1}{\partial h_1} \cdot \frac{\partial H_1}{\partial R_1}\right)$, and Case (A) is satisfied.

Taking Propositions 11 and 13 together, we have some conclusions. A strictly binding constraint (in the last period) is essential for us to be able to compare the equilibria of finite and infinite horizon models. If the infinite horizon model predicts a current capacity cycle, then there is a finite horizon model with an equilibrium which mimics the steady cycle. If a finite horizon model predicts that overflow binds in the final period, then (subject to some conditions) there is an infinite horizon model with a mimicking steady cycle. However, there are cases in which an infinite horizon model predicts an overflow cycle, while a finite horizon model predicts an equilibrium in which the only constraint which ever binds is the current capacity constraint.

5 Discussion

We have presented an infinite horizon model of competition between a hydro power plant and a thermal power plant. We have suggested steady cycles as the long term prediction within a Markov Prefect equilibrium of this model. Unlike many long run predictions, we can expect a steady cycle to be an accurate prediction within a finite time horizon. A steady cycle behaves in many ways like the subgame perfect equilibrium of a one year long finite horizon model. In both cases, it is typical for a constraint to bind in some (the last) period. Behavior in that period is nailed down by the need to satisfy the constraint. Behavior in other (earlier) periods is nailed down by using the Euler equation and knowledge about later periods.

There are limitations to the similarities between the equilibria of the finite horizon and infinite horizon models. There are steady cycles which are not representable, and there are subgame perfect equilibria of (year long) finite horizon models which are not valid. However, the one year models with which we draw comparison are typically not calibrated and then used to make accurate hour by hour predictions. In addition, a one year model is related to our model with a yearly cycle in exogenous variables in the same manner in which a one period model is related to the more standard dynamic game with stationary exogenous variables. With these facts in mind, it seems that the similarities between the equilibria in the two types of models are more striking than the differences.

One of the strengths of our paper as compared to many other infinite horizon models is that we don't assume perfect competition. A perfectly competitive firm acts to set $P_t = r \cdot P_{t+1}$. As compared to our Euler equation, this equation drops the strategic term, and sets Hydro's marginal revenue to the market price. Even ignoring the strategic term, Bushnell [3] finds using the marginal revenue much more accurate than using the price when modeling the California energy market.

In the name of tractability, we have included neither non-negativity nor turbine capacity constraints. An earlier version of this paper did include a non-negativity constraint for Hydro. We discuss below the implication from the inclusion of a non-negativity constraint. This sheds light on the implication for the inclusion of a turbine capacity constraint as well.

A period t non-negativity constraint requires that $h_t \geq 0$. Let δ_t denote the multiplier on this constraint. Hydro's Euler equation becomes

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t + \delta_t = r \left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{dH_{t+1}}{dR_{t+1}} + \delta_{t+1} \right)$$

If the period t non-negativity constraint were to bind, then $h_t = 0$ and $\delta_t \geq 0$ captures the difference between $\frac{\partial \pi_t}{\partial h_t}$ and the shadow cost of water.

Of greater interest, we see that the period $t+1$ non-negativity multipliers also appears in the new period t Euler. If non-negativity binds in period $t+1$, then $h_{t+1} = 0$

making $\frac{\partial \pi_{t+1}}{\partial q_{t+1}} = 0$. The period t Euler equation becomes

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t + \delta_t = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \delta_{t+1}\right)$$

It can be demonstrated that if non-negativity binds in period $t+1$, then $\kappa_{t+1} = \theta_{t+1} = 0$. Hence, the period $t+1$ Euler equation is

$$\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \delta_{t+1} = r\left(\frac{\partial \pi_{t+2}}{\partial h_{t+2}} + \frac{\partial \pi_{t+2}}{\partial q_{t+2}} \cdot \frac{\partial Q_{t+2}}{\partial h_{t+2}} \cdot \frac{dH_{t+2}}{dR_{t+2}} + \delta_{t+2}\right)$$

We can combine these equations to arrive at

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t + \delta_t = r^2\left(\frac{\partial \pi_{t+2}}{\partial h_{t+2}} + \frac{\partial \pi_{t+2}}{\partial q_{t+2}} \cdot \frac{\partial Q_{t+2}}{\partial h_{t+2}} \cdot \frac{dH_{t+2}}{dR_{t+2}} + \delta_{t+2}\right)$$

Proceeding in this manner, we see that if the first period following period t in which the non-negativity constraint does not bind is period $t+k$, then output in period t is determined by

$$\frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t + \delta_t = r^k\left(\frac{\partial \pi_{t+k}}{\partial h_{t+k}} + \frac{\partial \pi_{t+k}}{\partial q_{t+k}} \cdot \frac{\partial Q_{t+k}}{\partial h_{t+k}} \cdot \frac{dH_{t+k}}{dR_{t+k}}\right) \tag{10}$$

Using Equation 10 rather than the original Euler Equation 4 adds nothing substantive to our understanding.

However, including non-negativity comes at a considerable cost. Including non-negativity requires added notation to identify the first period following t in which output is non-zero. More importantly, if one includes non-negativity constraints, then there is the possibility that they bind weakly. Weakly binding non-negativity causes a kink in the strategy variable both in the current and (generically) at least one preceding period. These kinks may be dealt with, just as the kinks from weakly binding current capacity constraint were addressed. However, doing so comes at the cost of increased convolutions.

Essentially identical issues surround our decision not to include a turbine capacity constraint for Hydro. Again, if the turbine capacity constraint binds strictly in period $t+1$, then something like Euler Equation 10 gives a relationship between period t output and output in the next period in which capacity is not binding.[11] Again, weakly binding turbine capacity constraints can create kinks which we wish to avoid.

We have also ignored non-negativity constraints and plant capacity constraints for Thermal. Our main reason for doing this is that these constraints have not featured prominently in the one year models in the literature. If we were to include these constraints, then the main consequence would be to, occasionally, mitigate the importance of the strategic term in Hydro's Euler Equation. In particular, if one of these constraints is binding in period $t+1$, then $\frac{\partial Q_{t+1}}{\partial h_{t+1}} = 0$ and Hydro will act to balance discounted marginal revenues for period t and period $t+1$.

[11] In this case, however, we must rely upon the fact that if the turbine capacity constraint is strictly binding in period $t+1$, then $\frac{dH_{t+1}}{dR_{t+1}} = 0$.

A Appendix

We use either the general expression for, e.g., $\frac{\partial \pi_t}{\partial h_t}$ or the specific expression from the linear model as convenience dictates. We provide explicit expressions here. Thermal's policy is a standard Cournot reaction function: $Q_t(h_t, R_t) = \frac{a_t - c - bh_t}{2b_t + z}$. Hydro's marginal revenue is $\frac{\partial \pi_t}{\partial h_t} = a_t - b_t \cdot q_t - 2b_t \cdot h_t$. If we set $q_t = Q_t(h_t, R_t)$, Hydro's marginal revenue becomes

$$\frac{\partial \pi_t}{\partial h_t} = A_t - B_t \cdot h_t$$
$$\text{where} \quad A_t = \tfrac{b_t(a_t - c) + z \cdot a_t}{2b_t + z}$$
$$B_t = \tfrac{b_t(3b_t + 2z)}{2b_t + z}.$$

We need too an algebraic expression for the strategic term in the shadow value of water. Using our expression above for Q_t, we have $\frac{\partial Q_t}{\partial h_t} = \frac{-b_t}{2b_t + z}$. In addition, $\frac{\partial \pi_t}{\partial q_t} = \frac{\partial P_t}{\partial q_t} \cdot h_t = -b_t \cdot h_t$. Putting together we have

$$\frac{\partial \pi_t}{\partial q_t} \cdot \frac{\partial Q_t}{\partial h_t} = S_t \cdot h_t$$
$$\text{where } S_t = \frac{b_t^2}{2b_t + z}.$$

The Euler can be expressed as $A_t - B_t \cdot h_t = r(A_{t+1} - B_{t+1} \cdot h_{t+1} + S_{t+1} \cdot h_{t+1} \cdot \frac{dH_t}{dR_t})$.

We prove a more general version of Proposition 1 which does not assume that $\frac{dH_t}{dR_t}$ is defined. Let H_t^+ and H_t^- denote, respectively, the right-hand and left-hand derivatives of H_t with respect to R_t. Let $\overline{H}_t' = \max\{H_t^-, H_t^+\}$ and let $\underline{H}_t' = \min\{H_t^-, H_t^+\}$.

We take a brief aside here regarding our lack of a non-negativity constraint. We generally act as if $\frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \geq 0$. This amounts to an assumption that $h_{t+1} \geq 0$.

Since we do not have any non-negativity constraints, this assumption may be violated. However, the only consequence would be to switch the roles played by $\overline{H}'_t$ and $\underline{H}'_t$. In particular, if $h_{t+1} \geq 0$, then $\frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \overline{H}'_t \geq \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \underline{H}'_t$. If $h_{t+1} < 0$, then the opposite inequality holds.

Lemma 1 *Equation 11 is a generalized Euler Relation which holds along the path of play of a MPE.*

$$r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \underline{H}'_{t+1}\right) \leq \frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t \leq r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \overline{H}'_{t+1}\right) \tag{11}$$

Proof: Because left-hand and right-hand derivatives are determined by the defined derivative for marginally different argument values, we first work with the assumption that $\frac{dH_t}{dR_t}$ is defined. The first order condition for Hydro's objective, Equation 3, is

$$r\frac{dV_{t+1}}{dR_{t+1}} = \frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t. \tag{12}$$

Because H_t and Q_t solve the maximization problems,

$$V_t(R_t) = \pi_t(H_t, Q_t) + \kappa_t(R_t + w_t - H_t) + \theta_t(\bar{R} - R_t - w_t + H_t) + rV_{t+1}(R_t + w_t - H_t).$$

The envelope condition in R_t is

$$\frac{dV_t}{dR_t} = \frac{\partial \pi_t}{\partial h_t} \cdot \frac{dH_t}{dR_t} + \frac{\partial \pi_t}{\partial q_t} \cdot \frac{dQ_t}{dR_t} + \left(\kappa_t - \theta_t + r\frac{dV_{t+1}}{dR_{t+1}}\right)\left(1 - \frac{dH_t}{dR_t}\right) \tag{13}$$

Because $\frac{\partial Q_t}{\partial R_t} = 0$, we know that $\frac{dQ_t}{dR_t} = \frac{\partial Q_t}{\partial h_t} \cdot \frac{dh_t}{dR_t}$. We use this fact, and substitute

Equations 12 into Equation 13 to arrive at

$$\frac{dV_t}{dR_t} = \frac{\partial \pi_t}{\partial h_t} + \frac{\partial \pi_t}{\partial q_t} \cdot \frac{\partial Q_t}{\partial h_t} \cdot \frac{dH_t}{dR_t}. \tag{14}$$

If we replace $\frac{dH_t}{dR_t}$ with H_t^+ (resp. H_t^-) in the RHS of Equation 14, then we have an expression for V_t^+ (resp. V_t^-.) Expressions for $\overline{V}'_t = \max\{V_t^-, V_t^+\}$ and $\underline{V}'_t = \min\{V_t^-, V_t^+\}$ then follow immediately. Of course if $\underline{V}'_t < \overline{V}'_t$, then Hydro's first order condition becomes

$$r\underline{V}'_{t+1} \leq \frac{\partial \pi_t}{\partial h_t} - \kappa_t + \theta_t \leq r\overline{V}'_{t+1}. \tag{15}$$

The result then follows by using the appropriately modified versions of Equation 14 to replace $\underline{V}'_{t+1}$ and $\overline{V}'_{t+1}$. □

Proof of Proposition 1: This follows immediately from Lemma 1 since if $\frac{dH_t}{dR_t}$ is defined, then $H_{t+1}^+ = H_{t+1}^-$ and Equation 11 becomes Equation 4. □

Before proving Propositions 2 and 3, we provide some results to improve our understanding of the Euler Relation 11.

Lemma 2 *If neither constraint binds strictly in period t, then the Euler Relation for period t is*

$$r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right) \leq \frac{\partial \pi_t}{\partial h_t} \leq r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^-\right) \tag{16}$$

Proof: Since no constraint binds strictly, $\theta_t = \kappa_t = 0$, and Hydro has no desire to change h_t. If the first inequality did not hold, then Hydro would wish to decrease h_t and increase h_{t+1}. Likewise if the second inequality did not hold, then Hydro would wish to increase h_t and decrease h_{t+1}. □

Corollary 1 is a useful implication of Lemma 2.

Corollary 1 *Consider a MPE. If $H_t^+ > H_t^-$, then a constraint is strictly binding in period $t-1$.*

Lemma 3 *Consider a MPE path of play.*

Overflow is strictly binding in period t, if and only if Equation 17 holds.

Current capacity is strictly binding in period t, if and only if Equation 18 holds.

$$\frac{\partial \pi_t}{\partial h_t} < r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right) \tag{17}$$

$$\frac{\partial \pi_t}{\partial h_t} > r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^-\right) \tag{18}$$

Proof: Hydro wants to decrease h_t if and only if Equation 17 holds. Hence, if equation 17 holds, then Hydro has a strict desire to decrease h_t but is unable to do so. That is, overflow is strictly binding. On the other hand, if Equation 17 fails, then Hydro has no desire to decrease h_t which is to say that overflow is not strictly binding. Symmetrically, Hydro wants to increase h_t if and only if Equation 18 holds. When Equation 18 holds (resp. fails,) Hydro wants to (resp. does not want to) increase h_t, which is to say that current capacity is (resp. is not) strictly binding. □

We now prove Propositions 2 and 3. To this end, we establish an explicit relationship between $\frac{dH_t}{dR_t}$ and $\frac{dH_{t+1}}{dR_{t+1}}$. If no constraint binds in period t, and $\frac{dH_{t+1}}{dR_{t+1}}$ is well defined; then let $f_t(\cdot)$ denote the relationship between $\frac{dH_t}{dR_t}$ and $\frac{dH_{t+1}}{dR_{t+1}}$. That is $\frac{dH_t}{dR_t} = f_t\left(\frac{dH_{t+1}}{dR_{t+1}}\right)$. We show that

$$\frac{dH_t}{dR_t} = f_t\left(\frac{dH_{t+1}}{dR_{t+1}}\right) = \frac{\frac{dH_{t+1}}{dR_{t+1}} - \beta_t\left(\frac{dH_{t+1}}{dR_{t+1}}\right)^2}{\alpha_t + \frac{dH_{t+1}}{dR_{t+1}} - \beta_t\left(\frac{dH_{t+1}}{dR_{t+1}}\right)^2} \tag{19}$$

$$\text{where } \alpha_t = \left(\frac{b_t(3b_t + z)}{r \cdot b_{t+1}(3b_{t+1} + z)}\right)\left(\frac{2b_{t+1} + z}{2b_t + z}\right) \text{ and } \beta_t = \frac{b_{t+1}}{3b_{t+1} + 2z}$$

Lemma 4 *Consider an MPE path. If no constraint binds in period* t*, and* $\frac{dH_{t+1}}{dR_{t+1}}$ *is well defined; then Equation 19 holds.*

Proof: Within this proof, let us simplify notation by setting $H_t' = \frac{dH_t}{dR_t}$ and $t = 0$. Use of the explicit expression yields the following Euler

$$A_0 - B_0 \cdot h_0 = r(A_1 - B_1 \cdot h_1 + S_1 \cdot h_1 \cdot H_1').$$

A total differential in h_0 and R_0 yields

$$\frac{B_0}{r} dh_0 = B_1 dh_1 + S_1 H_1' dh_1 + S_1 h_1 H_1'' dR_1$$

Using $dR_1 = dR_0 - dh_0$, $dh_1 = H_1' dR_1$, and a bit of algebra yields

$$H_0' = \frac{r\left(-B_1 H_1' + S_1 (H_1')^2 + S_1 h_1 H_1''\right)}{-B_0 + r\left(-B_1 H_1' + S_1 (H_1')^2 + S_1 h_1 H_1''\right)}. \tag{20}$$

We see that the RHS of this equation is constant in R_0. Hence, H_0' does not depend directly upon R_0. Of course, this applies to all periods, so that we can set $H_1'' = 0$ in the above equation. If we then divide top and bottom by $r \cdot B_1$, we have

$$H_0' = \frac{H_1' - \left(\frac{S_1}{B_1}\right)(H_1')^2}{\frac{B_0}{r \cdot B_1} + H_1' + \left(\frac{S_1}{B_1}\right)(H_1')^2}.$$

Plugging in the values for B_t and S_t, and returning to the appropriate notation yields Equation 19. □

Lemma 5 *Let* $f_t(\cdot)$ *be defined by Equation 19 and let* $\bar{f}_t(\cdot) = f_t \circ f_{t+1} \circ \ldots \circ f_{t+D-1}(x_{t+D})$*.*
(A) $f_t(0) = 0$ *and* $f_t(1) < 1$*.*

(B) $f_t(\cdot)$ is a strictly increasing function on the range $[0,1]$.

(C) If $0 < x \leq 1$, then $\bar{f}_t(x) < r^D \cdot x$.

Proof: Let $g_t(x) = x - \beta_t x^2$. It follows that $f_t(x) = \frac{g_t(x)}{\alpha_t + g_t(x)}$. We note that $f_t'(x) = \frac{\alpha_t \cdot g_t'(x)}{[\alpha_t + g_t(x)]^2}$ and $f_t''(x) = \frac{\alpha_t \cdot [\alpha_t + g_t(x)] g_t''(x) - 2\alpha_t \cdot [g_t'(x)]^2}{[\alpha_t + g_t(x)]^3}$. We observe that $f_t(0) = g_t(0) = 0$ and $f_t(1) = \frac{1-\beta_t}{\alpha_t + 1 - \beta_t} < 1$, showing point (A). We see that f_t' has the same sign as $g_t' = 1 - 2\beta_t x$. Since $\beta_t < 1/2$, it follows that $f_t' > 0$ showing point (B). It remains only to show point (C). Since $g_t'' = -2\beta_t < 0$, it follows that $f_t'' < 0$. Since $\bar{f}_t(\cdot)$ is a composite function of strictly concave functions, it follows that $\bar{f}_t(\cdot)$ is strictly concave as well. We note that $\bar{f}_t'(x_{t+D}) = \prod_{i=t}^{t+D-1} \frac{a_i \cdot g_i'(x_i)}{[\alpha_i + g_i'(x_i)]^2}$. Hence $\bar{f}_t'(0) = \prod_{i=t}^{t+d-1} \frac{\alpha_i}{[\alpha_i]^2} = \prod_{i=t}^{t+d-1} \frac{1}{\alpha_i} = \prod_{i=0}^{T-1} \left(\frac{r \cdot b_{t+1}(3b_{t+1}+z)}{b_t(3b_t+z)} \right) \left(\frac{2b_t+z}{2b_{t+1}+z} \right) = r^D < 1$. Since $\bar{f}_t(\cdot)$ is increasing and strictly concave, it follows that $\bar{f}_t(x) < \bar{f}_t(0) + x \cdot \bar{f}_t'(0) = 0 + x \cdot r^D$. □

Proof of Proposition 2:

Point (C) follows immediately from the fact that R_t does not appear in Equation 19. Points (A) and (B) follow immediately from Points (A) and (B) of Lemma 5. □

Proof of Proposition 3:

Let $\bar{f}_t^y(\cdot) = f_t \circ f_{t+1} \circ \ldots \circ f_{t+yD-1}(\cdot)$ It is immediate that for $0 \leq x \leq 1$, $\bar{f}_t^y(x) \to 0$ as $y \to \infty$ from which the result follows. □

A.1 Kinks

We now consider the creation and propagation of the kinks in H_t which render $\frac{dH_t}{dR_t}$ undefined.

Observation 1 *Within a Markov Perfect Equilibrium, the following statements hold:*

(A) $0 \leq H_t^- \leq 1$ and $0 \leq H_t^+ \leq 1$.

(B) If current capacity is weakly binding in period t, then $H_t^- = 1$.

(C) If overflow is weakly binding in period t, then $H_t^+ = 1$.

The above observations follows from the fact that $\frac{dH_t}{dR_t}$ is defined on either side of a kink in H_t, and that H_t^+ and H_t^- are defined by the value of $\frac{dH_t}{dR_t}$ on the appropriate side of the kink.

We now investigate the extent to which $f_t(\cdot)$ can be extended to describe the relationship between H_t^+ and H_{t+1}^+ (resp. H_t^- and H_{t+1}^-).

Lemma 6 *Consider an MPE and a period t in which no constraint is strictly binding. (A) If overflow is not (weakly) binding and Equation 21 holds, then $H_t^+ = f_t(H_{t+1}^+)$. (B) If current capacity is not (weakly) binding and Equation 22 holds, then $H_t^- = f_t(H_{t+1}^-)$.*

$$\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right) \tag{21}$$

$$\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^-\right). \tag{22}$$

Proof: We first note that the assumptions in statement (A) (resp. (B)) assure that a constraint in period t does not determine H_t^+ (resp. H_t^-.) Consider a slight exogenous increase (resp. decrease) in R_t. If $H_t^+ = 1$ (resp. $H_t^- = 1$) then this would decrease (resp. increase) $\frac{\partial \pi_t}{\partial h_t}$ while having no impact on the RHS of Equation 21 (resp. Equation 22.) Given that Equation 21 (resp. Equation 22) holds prior to this change, this must lead to a violation of the first (resp. second) inequality from Equation 16. Hence $H_t^+ < 1$ (resp. $H_t^- < 1$.) This means that the exogenous increase (resp. decrease) in R_t leads to an increase (resp. decrease) in R_{t+1}. This moves us into a region in which no constraint binds (even weakly) in period t, and $\frac{dH_{t+1}}{dR_{t+1}}$ is defined and equal to the value of H_{t+1}^+ (resp. H_{t+1}^-) prior to the change in R_t. At this point Lemma 4 applies, and

$\frac{dH_t}{dR_t} = f_t\left(\frac{dH_{t+1}}{dR_{t+1}}\right)$, which is to say that $H_t^+ = f_t\left(H_{t+1}^+\right)$ (resp. $H_t^- = f_t\left(H_{t+1}^-\right)$.) $\square$

Of course if $H_{t+1}^+ < H_{t+1}^-$, then at least one of Equations 21 and 22 must fail. We now address that issue. For convenience, we restate the Euler Relation Equation 16:

$$r\left(\frac{\partial\pi_{t+1}}{\partial h_{t+1}} + \frac{\partial\pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right) \leq \frac{\partial\pi_t}{\partial h_t} \leq r\left(\frac{\partial\pi_{t+1}}{\partial h_{t+1}} + \frac{\partial\pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^-\right).$$

Lemma 7 *(A) If the first inequality in Equation 16 is strict, then* $H_t^+ = 1$.
(B) If the second inequality in Equation 16 is strict, then $H_t^- = 1$.

Proof: We first note that if either constraint binds strictly, then the result follows. Further, a weakly binding overflow (resp. current capacity) makes statement (A) (resp. statement (B)) true, and imposes no restriction on H_t^- (resp. H_t^+). We henceforth assume that no constraint binds in period t. Assume that $\frac{\partial\pi_t}{\partial h_t} < r\left(\frac{\partial\pi_{t+1}}{\partial h_{t+1}} + \frac{\partial\pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^-\right)$. In this case, the last unit of water used in period t is less profitable than the last unit of water passed to period $t+1$. Hence if there is an exogenous decrease in R_t, then that should be absorbed entirely within period t. That is, $H_t^- = 1$. If $\frac{\partial\pi_t}{\partial h_t} > r\left(\frac{\partial\pi_{t+1}}{\partial h_{t+1}} + \frac{\partial\pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right)$, then the next unit of water to be used in period t is more valuable than the next unit of water passed to period $t+1$. Hence, if there is an exogenous increase in R_t, then it should all be used in period t. That is, $H_t^+ = 1$.$\square$

Lemma 8 *Consider a MPE and periods* $t < \tau$.
(A) If $H_\tau^+ > 0$, *then* $H_t^+ > 0$.
(B) If $H_\tau^- > 0$, *then* $H_t^- > 0$.

Proof: If a constraint is strictly binding in period t, then the result follows automatically. If overflow (resp. current capacity) is weakly binding in period t, then statement (A) (resp. statement (B)) follows. Otherwise, an iterative application of Lemmas 6 and 7 prove the result.□

A.2 Steady Cycles

Lemma 9 *Consider a MPE path. If no constraint binds on or after some period τ, then $\exists\overline{\tau}$ such that $H_t^+ = 0$ and $h_t = \mathcal{C}_t$ for all $t \geq \overline{\tau}$.*

Proof: We first show that for t sufficiently large $H_t^+ = 0$ and $\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}}\right)$. We use the second fact to show that $h_t = \mathcal{C}_t$ for t sufficiently large.

A kink in H_t arises either because of a weakly binding constraint in period t or a kink in period H_{t+1}. Since no constraint binds for $t \geq \tau$, either $\frac{dH_t}{dR_t}$ is defined for all $t \geq \tau (= \overline{\tau})$, or $\exists\overline{\tau}$ such that H_t is kinked for all $t \geq \overline{\tau}$. In the first case, Proposition 3 implies that $H_t^+ = \frac{dH_t}{dR_t} = 0$ for all $t \geq \overline{\tau}$. If $\frac{dH_t}{dR_t} = 0$ for all $t \geq \tau$, then the Euler Equation 4 is $\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}}\right)$. Now consider the case in which H_t is kinked for $t \geq \overline{\tau}$. Since no constraint binds for $t \geq \overline{\tau} \geq \tau$, Corollary 1 implies that $H_t^+ < H_t^-$. By Lemmas 6 and 7, this implies that $H_t^- = 1$ and $H_t^+ = f_t(H_{t+1}^+)$. Let $\bar{f}_t^y = f_t \circ f_{t+1} \circ \ldots \circ f_{t+yD-1}(\cdot)$. It follows that $H_t^+ = \bar{f}_t^y(H_{t+yD}^+)$. By Lemma 5 this implies that $H_t^+ = 0$. In addition, by Lemmas 6 and 7, it must be the case that $\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H_{t+1}^+\right) = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}}\right)$.

Hence in either of the two possible cases, $\frac{\partial \pi_t}{\partial h_t} = r^k \frac{\partial \pi_{t+k}}{\partial h_{t+k}}$. That is, either $\frac{\partial \pi_{t+k}}{\partial h_{t+k}}$ becomes unbounded as k becomes large, or $\frac{\partial \pi_t}{\partial h_t} = \frac{\partial \pi_{t+k}}{\partial h_{t+k}} = 0$. However, the constraints on R_{t+k+1} assure that $\frac{\partial \pi_{t+k}}{\partial h_{t+k}}$ can't become unbounded. Hence $\frac{\partial \pi_t}{\partial h_t} = 0$ which means that $h_t = \mathcal{C}_t$. □

Proof of Proposition 4:

Let R_t^* and h_t^* denote the reservoir level and water use along the observed MPE, and let

$\hat{R}_d$ and $\hat{h}_d$ denote the reservoir level and water use within the steady cycle. If neither hydro constraint binds on a date d, then for t sufficiently large neither hydro constraint binds in period t with $d(t) = d$. Hence if the MPE converges to an open cycle, then there is some τ after which neither hydro constraint binds. From Lemma 9, this implies that $\exists \bar{\tau}$ such that $h_t = \mathcal{C}_t$ for $t \geq \bar{\tau}$. Hence $h_t = h_{t+D}$ for $t \geq \bar{\tau}$, which means that the MPE has converged by period $\bar{\tau}$.

Now consider the case in which the converged to steady cycle has a binding constraint. Clearly the convergence of R_t to $\hat{R}_{d(t)}$ implies the convergence of h_t^* to $\hat{h}_{d(t)}$. For the Euler Equations to hold for $h_t^* \approx \hat{h}_{d(t)}$ requires that H_t^+ and H_t^- along the MPE are approximately equal to their values in the steady cycle. This implies that for t sufficiently large, a constraint binds strictly (resp. weakly) in period t if and only if it binds strictly (resp. weakly) on date $d(t)$ within the steady cycle. This means that H_t^+ and H_t^- take exactly the values that they take within the steady cycle. The fact that the constraints bind as they do in the steady cycle means that hydro uses exactly $\overline{W}$ every year. This and the fact that the Eulers are linear and independent, implies that for t sufficiently large $h_t^* = \hat{h}_{d(t)}$. $\square$

Proof of Proposition 5:

By assumption, no constraint binds once the cycle has been entered. By Lemma 9, this implies that $h_t = \mathcal{C}_t$ once the cycle is entered. $\square$

Lemma 10 *If no constraint binds in period $t-1$, and overflow binds in period t; then* $\frac{dH_t}{dR_t} = 1$.

Proof: If H_t is kinked, then $1 = H_t^+ > H_t^-$. By Corollary 1 this implies that a hydro constraint binds strictly in period $t-1$. By this contradiction, H_t is not kinked. $\square$

Proof of Proposition 6:

If overflow binds weakly on a date d and H_d is kinked, then by Corollary 1, a hydro constraint binds strictly on date $d-1$. On the other hand, if an overflow binds weakly on date d, but H_d is not kinked, by Lemmas 6 and 7, it must be the case that $H^-_{d+1} > H^+_{d+1}$. This requires that there is some (weakly) future date on which current capacity is weakly binding. In neither case is there a pure weak overflow cycle. So say that we have a pure overflow cycle, and consider $\frac{dH_d}{dR_d}$. If overflow is strictly binding then $\frac{dH_d}{dR_d} = 1$. On the other hand, if starting from date d, the overflow binds strictly the next time that it binds, then Lemma 4 asserts that $\frac{dH_d}{dR_d}$ is defined (iteratively) by Equation 19. Since we found above that (absent any weakly binding current capacity constraints) a weakly binding overflow must be preceded by a strictly binding overflow, $\frac{dH_d}{dR_d}$ is defined on every date on which the overflow does not bind weakly. □

Lemma 11 *If a steady cycle is either a pure current capacity cycle or a pure weak current capacity cycle, then either:*

(1) strictly less than the Cournot output is produced in each period or

(2) exactly the Cournot output is produced in each period.

Proof: We first note that once the cycle has been entered, the overflow constraint never binds. By Lemma 3, this means that once the cycle has been entered

$$\frac{\partial \pi_t}{\partial h_t} \geq r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H^+_{t+1}\right). \tag{23}$$

If $\frac{\partial \pi_{t+1}}{\partial h_{t+1}} < 0$, then Equation 23 implies that $\frac{\partial \pi_t}{\partial h_t} > \frac{\partial \pi_{t+1}}{\partial h_{t+1}}$. On the other hand, if $\frac{\partial \pi_{t+1}}{\partial h_{t+1}} \geq 0$, then Equation 23 implies that $\frac{\partial \pi_t}{\partial h_t} \geq 0$. Say such that the cycle has been entered by period $t - D$. We first show that $h_t \leq \mathcal{C}_t$. If we assume otherwise, then $\frac{\partial \pi_t}{\partial h_t} < 0$. By

Equation 23, this implies that $\frac{\partial \pi_t}{\partial h_t} > \frac{\partial \pi_{t+1}}{\partial h_{t+1}}$. On the other hand, iterating Equation 23 backwards through time implies that $\frac{\partial \pi_t}{\partial h_t} < \frac{\partial \pi_{t+1-D}}{\partial h_{t+1-D}}$. Since $d(t+1) = d(t-D+1)$, this means that $\frac{\partial \pi_t}{\partial h_t} < \frac{\partial \pi_{t+1}}{\partial h_{t+1}}$. By this contradiction, $\frac{\partial \pi_t}{\partial h_t} \geq 0$ and $h_t \leq \mathcal{C}_t$ once the steady cycle has been entered. Now let us say that the current capacity constraint binds (possibly weakly) in period τ. Consider first the case that $h_\tau < \mathcal{C}_\tau$. In this case, $\frac{\partial \pi_\tau}{\partial h_\tau} > 0$, and Equation 23 implies that $h_t < \mathcal{C}_t$ in every period of the cycle. Now consider the case in which $h_\tau = \mathcal{C}_\tau$. The period τ version of Equation 23 is

$$0 \geq r \left(\frac{\partial \pi_{\tau+1}}{\partial h_{\tau+1}} + \frac{\partial \pi_{\tau+1}}{\partial q_{\tau+1}} \cdot \frac{\partial Q_{\tau+1}}{\partial h_{\tau+1}} \cdot H^+_{\tau+1} \right).$$

Since $\frac{\partial \pi_{\tau+1}}{\partial h_{\tau+1}} \geq 0$, the this equation can hold only if $0 = \frac{\partial \pi_{\tau+1}}{\partial h_{\tau+1}} = H^+_{\tau+1}$. Iterating this argument forward, we see that $\frac{\partial \pi_t}{\partial h_t} = 0$ for each t once the cycle has been entered. □

Proof of Proposition 7:

Assume that the Proposition is false. Lemma 11 then implies that $h_t = \mathcal{C}_t$ within the steady cycle. This means that $\frac{\partial \pi_t}{\partial h_t} = 0 = \frac{\partial \pi_{t+1}}{\partial h_{t+1}}$ within the steady cycle. Consider a period $t+1$ on which the current capacity constraint binds strictly. We know that $\frac{dH_{t+1}}{dR_{t+1}} = 1$. Hence, the period t Euler equation is $-\kappa_t = r \left(\frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \right) > 0$. By this contradiction, it is not possible for every $h_t = \mathcal{C}_t$. An application of Lemma 11 then proves the result. □

Proof of Proposition 10:

Since we are looking at a pure (weak) current capacity cycle, it follows that Equation 23 holds. We again simplify notation and let $\pi'_t = \frac{\partial \pi_t}{\partial h_t}$, and $S_t \cdot h_t = \frac{\partial \pi_t}{\partial q_t} \cdot \frac{\partial Q_t}{\partial h_t} = \frac{b_t^2 \cdot h_t}{2b_1 + z}$. Let t be a period in the cycle such that the current capacity constraint binds in period

$t-1$, but not in period t. Equation 23 implies that

$$\pi'_t \geq \left(\sum_{j=1}^{D-1} r^j S_{t+j} \cdot h_{t+j} \cdot H^+_{t+j} \right) + r^{D-1} \left(\pi'_{t+D-1} \right).$$

By Proposition 2 and Lemmas 6 and 7, there are only a finite number of values each H^+_{t+j} can take within a steady cycle. Hence, there is minimum value that $\sum_{j=1}^{D-1} r^j S_{t+j} \cdot h_{t+j} \cdot H^+_{t+j}$ can take if all inflows of water are used. Further, this value must be strictly greater than zero. Hence, we have that $\pi'_t \geq K + r^{D-1}\left(\pi'_{t+D-1}\right)$ where $K > 0$. In addition, since the current capacity constraint binds in period $t+D-1$, it follows that $\pi'_{t+D-1} \geq r(\pi'_{t+D} + S_{t+D} \cdot h_{t+D} \cdot H^+_{t+D}) = r(\pi'_t + S_t \cdot h_t \cdot H^+_t)$. Clearly there is a $\bar{r}$ above which at least one inequality must fail. □

A.3 Comparison to Finite Horizon

Lemma 12 *Let the Large Reservoir Assumption hold.*

If current capacity binds in period t then the overflow does not bind in period $t+1$.

Proof Let us work with $t=0$. Assume that current capacity binds in $t=0$ and overflow binds in period $t=1$. By Lemma 3, $A_0 - B_0 h_0 > r(A_1 - (B_1 - S_1)h_1)$. Clearly $\overline{P}_t = A_t$. In addition, since current capacity binds, we know that $h_0 \geq 0$. Hence, we know that $\overline{P}_0/r > \overline{P}_1 - (B_1 - S_1)h_1$ or

$$-h_1 < (\overline{P}_0/r - \overline{P}_1) \cdot \frac{2b_1 + z}{2b_1^2 + 2zb_1} \leq \frac{\overline{P}_0 - r\overline{P}_1}{rb_1} \leq \max\left\{\frac{\overline{P}_0 - r\overline{P}_1}{rb_1}, 0\right\} \equiv \rho_1.$$

$R_2 = R_1 - h_1 + w_1 < 0 + \rho_1 + w_1 < \overline{R}$ by the Large Reservoir Assumption. Hence overflow does not bind in period 1, by which contradiction the Lemma is proven. □

Let $\mathcal{H}_\gamma^+$ and $\mathcal{H}_\gamma^-$ denote the right-hand and left-hand derivatives of H_γ^F with respect to R_γ^F. Let $\underline{\mathcal{H}}'_\gamma = \min\{\mathcal{H}_\gamma^+, \mathcal{H}_\gamma^-\}$ and $\overline{\mathcal{H}}'_\gamma = \max\{\mathcal{H}_\gamma^+, \mathcal{H}_\gamma^-\}$.

Lemma 13 *Consider a MPE which converges to a steady cycle prior to period $\tau - D$ and a finite horizon model which is a run ending on date $d(\tau)$. If $\mathcal{H}_{\bar\gamma}^+ = H_{\tau+\bar\gamma}^+$ and $\mathcal{H}_{\bar\gamma}^- = H_{\tau+\bar\gamma}^-$, and the equilibria of the two models mimic each other; then $\mathcal{H}_\gamma^+ = H_{\tau+\gamma}^+$ and $\mathcal{H}_\gamma^- = H_{\tau+\gamma}^-$ for $1 \le \gamma < \bar\gamma$.*

Proof: The period $\gamma < D^F$ first order condition in the finite horizon model is identical to the period $\tau + \gamma$ Euler condition. Hence applying Lemmas 6 and 7 determine the same relationship between $H_{\tau+\gamma}$ and $H_{\tau+\gamma+1}$ as they define between H_τ and $H_{\tau+1}$. Hence, the result follows from iterative applications of Lemmas 6 and 7. □

Lemma 14 *Consider a MPE which converges to a steady cycle prior to period $\tau - D$ and a finite horizon model which is a run ending on date $d(\tau)$. Assume that the finite horizon model has an equilibrium which mimics the MPE.*
(A) A constraint binds in period γ of the finite horizon model if and only if it binds in period $\tau + \gamma$ of the infinite horizon model.
(B) If $\gamma < D^F$, $H_{\tau+\gamma}^+ = \mathcal{H}_\gamma^+$, and $H_{\tau+\gamma}^- = \mathcal{H}_\gamma^-$, then a constraint binds strictly in period $\gamma - 1$ of the finite horizon model if and only if it binds strictly in period $\tau + \gamma - 1$ of the infinite horizon model.

Proof: The first statement follows immediately from the fact that $\{h_{\tau+\gamma}, R_{\tau+\gamma}\} = \{h_\gamma^F, R_\gamma^F\}$. Given the first statement, the second statement follows from Lemma 13 and the fact that the $\gamma < D^F$ first order condition is identical to the $\tau + \gamma$ Euler.□

We refer to a Finite Horizon Model subgame perfect Equilibrium as a FHME.

Lemma 15 *Consider a FHME.*

(1) If $h^F_D = \mathcal{C}^F_D$ *and overflow binds strictly on* $\gamma = D^F - 1$*, then* $h^F_{D-1} \geq \mathcal{C}^F_{D-1}$*. This inequality is strict if overflow is not weakly binding in period* $\gamma = D^F$*.*

(2) If If $h^F_D = \mathcal{C}^F_D$ *and current capacity binds strictly on* $\gamma = D^F - 1$*, then* $h^F_{D-1} < \mathcal{C}^F_{D-1}$*.*

Proof: If $h^F_D = \mathcal{C}^F_D$, then no constraint binds strictly in period $\gamma = D^F$. If current capacity does not bind weakly, then $\mathcal{H}^+_D = 0$. If overflow does not bind weakly, then $\mathcal{H}^-_D = 0$. Since it is not possible for both constraints to bind, we have $\underline{\mathcal{H}}_D = 0$. If overflow is strictly binding in period $D^F - 1$, then by Lemma 3 $\frac{\partial \pi^F_{D-1}}{\partial h^F_{D-1}} \leq r\left(\frac{\partial \pi^F_D}{\partial h^F_D} + \frac{\partial \pi^F_D}{\partial q^F_D} \cdot \frac{\partial Q_D}{\partial h^F_D} \cdot \underline{\mathcal{H}}_D\right) = 0$. Further, if overflow is not weakly binding in period D^F, then $\mathcal{H}^+_D = \mathcal{H}^-_D = 0$, and we can replace the weak inequality with a strict inequality. This shows statement (1). If current capacity is strictly binding in period $D^F - 1$, then by Lemma 3, $\frac{\partial \pi^F_{D-1}}{\partial h^F_{D-1}} > r\left(\frac{\partial \pi^F_D}{\partial h^F_D} + \frac{\partial \pi^F_D}{\partial q^F_D} \cdot \frac{\partial Q_D}{\partial h^F_D} \cdot \mathcal{H}^-_D\right) \geq 0$ which shows statement (2). □

Let us say that a constraint *binds cleanly* in period t if $\frac{dH_t}{dR_t} = 1$. If a constraint binds in period t and H_t is kinked, then we say that the constraint *binds with kink.* We use the same expressions for the finite horizon model. Obviously, if a constraint binds with kink, then it binds weakly. The converse is not true.

Lemma 16 *Consider a MPE. (A) Overflow binds both cleanly and weakly in period* t *if and only if Equation 24 holds.*

(B) Current capacity binds both cleanly and weakly in period t *if and only if Equation 25 holds.*

$$r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H^+_{t+1}\right) \leq \frac{\partial \pi_t}{\partial h_t} < r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot H^-_{t+1}\right) \quad (24)$$

$$r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}}+\frac{\partial \pi_{t+1}}{\partial q_{t+1}}\cdot\frac{\partial Q_{t+1}}{\partial h_{t+1}}\cdot H^{+}_{t+1}\right)<\frac{\partial \pi_t}{\partial h_t}\leq r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}}+\frac{\partial \pi_{t+1}}{\partial q_{t+1}}\cdot\frac{\partial Q_{t+1}}{\partial h_{t+1}}\cdot H^{-}_{t+1}\right) \quad (25)$$

Proof: Case (A) (resp. (B)): if overflow (current capacity) binds weakly, then $H_t^+ = 1$ (resp $H_t^- = 1$.) Lemmas 6 and 7 make clear that $H_t^- = 1$ (resp. $H_t^+ = 1$) holds in addition if and only if the strict inequality in Equation 24 (resp. Equation 25) holds. Finally if the weak inequality in Equation 24 (resp. Equation 25) failed, then by Lemma 3 overflow (resp. current capacity) would bind strictly. □

Lemma 17 *Let the Large Reservoir Assumption hold. Consider a MPE which converges to a steady cycle prior to period τ and a finite horizon model which is a run ending on date $d(\tau)$. Assume that the finite horizon model has an equilibrium which mimics the steady cycle.*
If current capacity binds cleanly (resp. with kink) in period τ of the MPE and binds with kink (resp. cleanly) in period D of the FHME, then the current capacity constraint binds strictly in both period $D-2$ of the FHME and period $\tau-2$ of the MPE. Furthermore Equation 26 (resp. Equation 27) holds and $h_{\tau-2} = h^F_{D-2} < \mathcal{C}_{\tau-2} = \mathcal{C}^F_{D-2}$.

Proof: $\mathcal{C}_{\tau-2} = \mathcal{C}^F_{D-2}$ follows because the finite horizon model is a run ending on date $d(\tau)$, and $h_{\tau-2} = h^F_{D-2}$ follows because the two equilibria mimic each other. We next note that in either case considered, $h^F_D \leq \mathcal{C}^F_D$. Now say that current capacity binds cleanly in period $\gamma = D^F$ of the FHME, and binds with kink in period τ of the MPE. It follows that $\frac{dH^F_D}{dR^F_D} = 1$, and $H^+_\tau < 1 = H^-_\tau$ Because $h_\tau = h^F_D$ and $h_{\tau-1} = h^F_{D-1}$, it follows that

$$r\left(\frac{\partial \pi_\tau}{\partial h_\tau}+\frac{\partial \pi_\tau}{\partial q_\tau}\cdot\frac{\partial Q_\tau}{\partial h_\tau}\cdot H^{+}_{\tau}\right)<\frac{\partial \pi_{\tau-1}}{\partial h_{\tau-1}}= r\left(\frac{\partial \pi_\tau}{\partial h_\tau}+\frac{\partial \pi_\tau}{\partial q_\tau}\cdot\frac{\partial Q_\tau}{\partial h_\tau}\cdot H^{-}_{\tau}\right). \quad (26)$$

Since $h_\tau \leq \mathcal{C}_\tau$, Equation 26 implies that $h_{\tau-1} < \mathcal{C}_{\tau-1}$. From Lemmas 6 and 7, Equation 26 implies that $H^+_{\tau-1} = 1 > H^-_{\tau-1}$. By Corollary 1, this implies that a constraint is strictly binding in period $t = \tau - 2$. Since $h_{\tau-1} + h_\tau < \mathcal{C}_{\tau-1} + \mathcal{C}_\tau$, Assumption 1 implies it is current capacity that binds. Since $h_{\tau-1} < \mathcal{C}_{\tau-1}$, the LHS of the period $t = \tau - 2$ Euler is strictly positive, which implies that $h_{\tau-2} < \mathcal{C}_{\tau-2}$.

The case in which current capacity binds with kink in period $\gamma = D$ and bind cleanly in period $t = \tau$ is exactly symmetric. In particular,

$$r\left(\frac{\partial \pi^F_D}{\partial h^F_D} + \frac{\partial \pi^F_D}{\partial q^F_D} \cdot \frac{\partial Q^F_D}{\partial h^F_D} \cdot \mathcal{H}^+_D\right) < \frac{\partial \pi^F_{D-1}}{\partial h^F_{D-1}} = r\left(\frac{\partial \pi^F_D}{\partial h^F_D} + \frac{\partial \pi^F_D}{\partial q^F_D} \cdot \frac{\partial Q^F_D}{\partial h^F_D} \cdot \mathcal{H}^-_D\right). \tag{27}$$

Essentially identical arguments then lead us to the conclusion that current capacity is strictly binding in $t = \tau - 2$ and $\gamma = D^F - 2$ with $h^F_{D-2} < \mathcal{C}^F_{D-2}$ $\square$

Lemma 18 *Let the large reservoir assumption hold, and consider a MPE which converges to a steady cycle prior to period $T - D$. This steady cycle is representable only if one of the following conditions holds.*

(A) $\exists \tau \geq T$ such that overflow binds cleanly in period τ, and $h_\tau > \mathcal{C}_\tau$.

(B) $\exists \tau \geq T$ such that current capacity binds cleanly in period τ, and $h_\tau < \mathcal{C}_\tau$.

(C) $\exists \tau, \chi \geq T$ such that the current capacity constraint binds weakly in period τ, $h_\tau = \mathcal{C}_\tau$, and overflow binds in period χ with $\frac{dH_\chi}{dR_\chi} = 1$.

(D) $\exists \tau \geq T$ such that $h_\tau \geq \mathcal{C}_\tau$, overflow binds weakly in period τ, and overflow binds strictly in period $\tau - 1$.

(E) $h_d = \mathcal{C}_d$ on each date d once the steady cycle has begun.

Proof: Let us say that the finite horizon model is a run ending on date $d(\psi)$, and set $\{h^F_\gamma, R^F_\gamma\} = \{h_{\psi+\gamma}, R_{\psi+\gamma}\}$. We run through the possibilities for period D of the finite horizon model. If the overflow constraint is strictly binding, then $\frac{dH_D^F}{dR_D^F} = 1$ and $\frac{\partial \pi_D^F}{\partial h_D^F} < 0$ ($\Leftrightarrow h_\tau > \mathcal{C}_\tau$). If $\frac{dH_\psi}{dR_\psi} = 1$, then we are in case (A) with $\tau = \psi$. If instead, there is a kink in H_ψ, then a constraint must be strictly binding in period $\psi - 1$. By the Large Reservoir Assumption and Lemma 12 it is overflow which binds and and we are in case (D) with $\tau = \psi$. If current capacity is strictly binding in period $\gamma = D^F$, then $\frac{dH_D^F}{dR_D^F} = 1$ and $\frac{\partial \pi_D^F}{\partial h_D^F} > 0$ ($\Leftrightarrow h_\tau < \mathcal{C}_\tau$). If $\frac{dH_\psi}{dR_\psi} = 1$, then we are in case (B) with $\tau = \psi$. If current capacity binds with kink in period $t = \psi$, then Lemma 17 implies that we are in case (B) with $\tau = \psi - 2$.

In the remaining cases, no constraint is strictly binding in period $\gamma = D^F$, and $h_D^F = \mathcal{C}_D^F$. By Lemma 15, if overflow is strictly binding in period $\gamma = D - 1$, then we are in either case (A) with $\tau = \psi - 1$ or case (D) with $\tau = \psi$. By Lemma 15, if current capacity is strictly binding in period $\gamma = D - 1$, then we are in case (B) with $\tau = \psi - 1$. We are left with cases in which no constraint is strictly binding in either period $\gamma = D^F - 1$ or $\gamma = D^F$. By Lemma 10, this means that overflow is not weakly binding in period $\gamma = D^F$.

So let us say now that current capacity binds weakly in period $\gamma = D^F$. If the current capacity constraint binds weakly in period $t = \psi$, then by Lemma 11 either there is a period $\xi > \psi$ in which the overflow constraint binds, or we are in case (E). If we are in the case in which the overflow binds, then either H_ξ is unkinked or it is preceded by a period in which the overflow binds strictly. Hence, in this event we are in case (C). If, on the other hand, current capacity binds strictly in period ψ, then an application of Lemma 17 implies that we are in case (B) with $\tau = \psi - 2$.

The last case is when no constraint binds in period $\gamma = D^F$ and no constraint binds strictly in period $\gamma = D^F - 1$. In this case, $h_D^F = \mathcal{C}_D^F$, and $\frac{dH_D^F}{dR_D^F} = 0$. The period $\gamma = D^F - 1$ FOC then implies that $h_{D-1}^F = \mathcal{C}_{D-1}^F$. Since the equilibria mimic each other: $h_\psi = \mathcal{C}_\psi$, $h_{\psi-1} = \mathcal{C}_{\psi-1}$, and no constraint binds strictly in period $\psi - 1$. This is only possible if $H_\psi^+ = 0$. By Lemma 8, this implies that within the steady cycle the current capacity does not bind strictly, and the overflow does not bind (weakly or strictly.) By Lemma 11, we must be in case (E). □

A.4 Generic Equilibria

In moving from Lemma 18 to Proposition 11 we need to introduce some terminology. The dates of a year form a circular number system. When speaking of dates, we follow the natural convention in which, for example, $D + 1 = 1$. Consider a steady cycle in which a constraint binds on date $\underline{d} - 1$ and $\overline{d}$, but on no dates in between. We say that $\beta = \{\underline{d}, ...\overline{d}\}$ forms a *block*. Notice that both $\overline{d} = \underline{d}$ and $\overline{d} = \underline{d} - 1$ are possible. That is, a block may include anything from a single date to the entire year. We denote the total inflow during block β by

$$\hat{W}(\beta) = \sum_{d=\underline{d}}^{\overline{d}} w_d.$$

For a block $\beta = \{\underline{d}, ...\overline{d}\}$, if current capacity binds with kink on date $\overline{d}$, then it is useful to break the block into two sub-blocks. We know $H_{\overline{d}}^+ < H_{\overline{d}}^-$. For ease of reference, we restate the Euler relation 16 for date a d in which no constraint binds;

$$r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H_{d+1}^+\right) \leq \frac{\partial \pi_d}{\partial h_d} \leq r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H_{d+1}^-\right).$$

There are three ways this Euler can hold. If $\frac{\partial \pi_d}{\partial h_d} = r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H_{d+1}^-\right)$, then we have $H_d^- < H_d^+ = 1$ and a constraint must bind strictly on date $d-1$ (i.e. $d = \underline{d}$.) If $\frac{\partial \pi_d}{\partial h_d} = r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H_{d+1}^+\right)$, then $H_d^+ < H_d^- = 1$, and H_d is kinked exactly as H_{d+1} is. Finally, if both inequalities in Euler Relation 16 are strict, then $\frac{dH_{\hat{d}}}{dR_{\hat{d}}} = 1$, and we denote this date as $\hat{d}(\beta)$ Sub-block one consists of $\{\underline{d}+1, ...\hat{d}(\beta)\}$, the dates within the block on which $\frac{dH_d}{dR_d}$ is defined. Sub-block 2 consists of $\{\hat{d}(\beta)+1, ...\underline{d}\}$, the dates within the block on which H_d is kinked. We call date $\hat{d}(\beta)$ the *dividing date*. The first sub-block may be empty, but because current capacity binds with kink on date $\overline{d}$, the second sub-block is not. We follow the convention of setting $\hat{d}(\beta) = 0$ if the first sub-block is empty.

It is useful to provide a notion for a generic equilibrium which does not include validity or representability. We say that a steady cycle (resp. FHME) is generic if $\exists\epsilon > 0$ such that every ϵ perturbation has a congruent steady cycle (resp. FHME.). Clearly if a steady cycle (resp. FHME) is generically representable (resp. generically valid) then both it and the equilibrium which mimics it are generic. Genericity is tied to the ability to change inflows slightly on a given date d, or block of dates including d, while maintaining a congruent steady cycle. We observe that solving for each h_d involves solving a system of linear equations subject to some linear constraints. The linear constraints are satisfied if we have correctly identified the dates on which Hydro's reservoir constraints bind. Since we are investigating genericity of an already identified steady cycle, there is no loss in focusing on only the system of linear equations. If $\frac{dH_{d+1}}{dR_{d+1}}$ is defined, and no constraint is strictly binding on date d, then Equation 5 is a linear equation between h_d and h_{d+1}. On the other hand, if no constraint is strictly binding on date d, but H_{d+1} has a kink, then the Relation 16 holds. If one of the weak inequalities in Relation 16 is in fact an equality, then again there is a linear equation relating h_d and h_{d+1}. If, instead, both inequalities are strict, then Relation 16 imposes a pair of linear constraints, but no linear equations. Finally, if a (reservoir) constraint binds, either weakly or strictly, on date $\overline{d}$, then this imposes a linear restriction on the total output for the block $\beta = \{\underline{d}, ...\overline{d}\}$

$$\sum_{d=\underline{d}}^{\overline{d}} h_d = \left(\sum_{d=\underline{d}}^{\overline{d}} w_d \right) + (1_{\text{cc-of}} - 1_{\text{of-cc}})\bar{R}. \tag{28}$$

The indicator variable $1_{\text{cc-of}}$ takes a value of one if current capacity binds on date $\underline{d} - 1$ and overflow binds on date $\overline{d}$. Otherwise it is equal to zero. The indicator variable $1_{\text{of-cc}}$ takes the value one if overflow binds on date $\underline{d} - 1$ and current capacity binds on date $\overline{d}$ and is zero otherwise. If both indicators equal zero, then we have a requirement that from $\underline{d}$ to $\overline{d}$ total outflows equal total inflows $\hat{W}(\beta)$. Hence, the reservoir goes from full (resp. empty) to full (resp. empty.) If one of the indicators is equal to one, then the reservoir goes from empty to full or vice versa. Hence we need to either subtract or add $\bar{R}$ to the amount of water used. Equation 28 ties the total water used in any given block to the inflows for that block. The other linear equalities tie the use on one date to the use on the next.

Consequently, a steady cycle in which constraints bind strictly or not at all has: one strict inequality for each binding constraint, and one linear equation for each unknown h_d. In this case, if we, for example, slightly increase inflow within a block, then we can satisfy all the required linear equalities by slightly increasing output on every date within the block. If the increase is small enough, then it won't upset any of the strict inequalities. Hence these types of steady cycles are generic.

On the other hand, if $\frac{dH_{d+1}}{dR_{d+1}}$ is defined and a constraint binds weakly in period d, then both Equation 28 and Equation 5 must hold. This can result in more equations than unknowns, which places a linear restriction on inflows and a failure of genericity. Finally, if date d is a dividing date on which no constraint binds, then date d does not impose an additional linear equation. Of course, we only have a dividing date if current capacity binds with kink on the last date of the block. Hence, generic steady cycles with weakly binding current capacity constraints are possible, but generic steady cycles with weakly binding overflow constraints are not.

Our next step is to determine what h_d depends upon within a steady cycle. Obviously, h_d must depend upon cost parameters and the demand parameter for that date. It will likely depend upon demand parameters for different dates as well. We take all this as given, and focus on what other things we need to known in order to solve for h_d. Hence if we write, for example, "h_d is a linear function of only $\hat{W}(\beta)$ and $h_{d+\Delta}$," then this means that we can solve for h_d without knowledge of inflows other than $\hat{W}(\beta)$ or outputs other than $h_{d+\Delta}$.

Lemma 19 *Consider a block $\beta = \{\underline{d}, ...\overline{d}\}$ of a steady cycle. If a constraint binds cleanly on date $\overline{d}$, then for each $d \in \beta$, h_d can be solved as a linear function of only $\hat{W}(\beta)$.*

Proof: Because the date $\overline{d}$ constraint binds strictly, we know that the Euler 5 holds in all earlier dates in the block. In addition Equation 28 must hold. This gives us $\overline{d}-\underline{d}+1$ linear and independent equations in $\overline{d}-\underline{d}+1$ unknowns. Further, these equations depends upon only $\hat{W}(\beta)$ and $\{h_d\}_{d \in \beta}$. □

Lemma 20 *Consider a block $\beta = \{\underline{d}, ...\overline{d}\}$ of a steady cycle. Assume that current capacity binds with kink on date $\overline{d}$.*
(A) If d is in the second sub-block of this block, then h_d can be solved for as a linear function of only $h_{\overline{d}+1}$.
(B) If d is in the first sub-block of this block, then h_d can be solved for as a linear function of only $h_{\overline{d}+1}$ and $\hat{W}(\beta)$.

Proof: For d in in the second block, $\frac{\partial \pi_d}{\partial h_d} = r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H^+_{d+1}\right)$ (unless $\hat{d} = 0$ and $d = \underline{d}$ in which case it is also possible that $\frac{\partial \pi_d}{\partial h_d} = r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H^-_{d+1}\right)$.) Hence, for each d in the second block, h_d is a linear function of only h_{d+1} which establishes the first statement. With the outputs in the second sub-block nailed down, the Eulers in the first $\hat{d} - \underline{d}$ dates combine with Equation 28 to nail down h_d in the first sub-block. Of course, Equation 28 depends upon both $\hat{W}(\beta)$ and the outputs in the second block, which demonstrates the second statement. □

Lemma 21 *Consider a steady cycle with blocks $\beta = 1, 2, ..N$ each ending on date $\overline{d}(\beta)$ with block $\beta + 1$ starting on date $\underline{d}(\beta + 1) = \overline{d}(\beta) + 1$. If a constraint binds cleanly on date $\overline{d}(N)$ and weakly on date $\overline{d}(1)$, then $h_{\overline{d}(1)}$ can be solved for as a function of only $\hat{W}(2), \hat{W}(3), ...\hat{W}(N)$.*

Proof: By Lemmas 19 and 20, if $1 < \beta < N$ and $h_{\overline{d}(\beta)+1}$ can be solved for as a linear function of only $\hat{W}(\beta + 1)$, $\hat{W}(\beta + 2), ...\hat{W}(N)$, then $h_{\overline{d}(\beta-1)+1}$ can be solved for as a linear function of only $\hat{W}(\beta)$, $\hat{W}(\beta + 1), ...\hat{W}(N)$. By Lemma 19 $h_{\overline{d}(N-1)+1}$ is a linear function of only $\hat{W}(N)$. Induction proves the result. □

Lemma 22 *Let the Large Reservoir Assumption hold. A steady cycle with a weakly binding overflow constraint is not generic.*

Proof: Consider a steady cycle containing a block which ends with a weakly binding overflow constraint. Denote this as block 1 ending on date $\overline{d}(1)$. Enumerate the remaining blocks β so that they end on date $\overline{d}(\beta)$ and block $\beta + 1$ starts on date $\overline{d}(\beta) + 1$. Let N denote the lowest number such that a constraint binds cleanly on date $\overline{d}(N)$. By Lemma 21, $h_{\overline{d}(1)}$ can be solved as a function of only $\hat{W}(2), ...\hat{W}(N)$. However, overflow must bind strictly on date $\overline{d}(1) - 1$, which means that $h_{\overline{d}(1)} = w_{\overline{d}(1)}$. Hence, $w_{\overline{d}(1)}$ is a linear function of $\hat{W}(2), ...\hat{W}(N)$. Clearly then, adding (or subtracting) some small ϵ to $w_{\overline{d}(1)}$ yields a perturbation for which there is no congruent steady cycle.

Our arguments above might seem to ignore the possibility of a congruent equilibrium which differs in that some blocks have different dividing dates. However, there are only a finite number of values each dividing date can take. Hence, one of a finite number of linear relationships must hold between $w_{\overline{d}(1)}$ and $\hat{W}(2), ...\hat{W}(N)$. Any which hold for the original set of inflow, won't hold after we have added ϵ to $w_{\overline{d}(1)}$. Any which don't hold with the original inflow provide at most a single value of ϵ which might be added to $w_{\overline{d}(1)}$ and yield a congruent steady cycle. □

Lemma 23 *A steady cycle with a current capacity constraint that binds weakly but cleanly is not generic.*

Proof: If current capacity binds on date d cleanly but weakly, then H_{d+1} is kinked (Lemma 16.) The block following date d must end with a weakly binding current capacity, and have an empty sub-block 1. From Lemmas 20 and 21, we know that every output in this block is determined by a linear relationship with the inflows from other blocks. In addition, within this block we have $\frac{\partial \pi_d}{\partial h_d} = r\left(\frac{\partial \pi_{d+1}}{\partial h_{d+1}} + \frac{\partial \pi_{d+1}}{\partial q_{d+1}} \cdot \frac{\partial Q_{d+1}}{\partial h_{d+1}} \cdot H^{+}_{d+1}\right)$. Hence it is not possible to increase h_d without also increasing h_{d+1}. Hence, if ϵ is added to the inflows for this block, then the weakly binding constraint at the end must cease to bind. □

Proof of Proposition 11: We first observe that from Lemma 22 and 23, if a constraint binds cleanly in a generic steady cycle, then it binds strictly. Hence, we need only rule out cases (C), (D), and (E) of Lemma 18 as not generically representable. Lemma 22 rules out Case (D). Case (E) is non-generic because it requires $\overline{W} = \overline{\mathcal{C}}$.

For case (C), set $\overline{d}(1) = d(\tau)$. Enumerate the remaining blocks β so that they end on date $\overline{d}(\beta)$ and block $\beta+1$ starts on date $\overline{d}(\beta)+1$. Let N denote the lowest number such that a constraint binds cleanly on date $\overline{d}(N)$. By Lemma 21, $h_{\overline{d}(1)}$ can be solved for as a function of $\hat{W}(2),...\hat{W}(N)$. However, we know that $h_{\overline{d}(1)} = \mathcal{C}_\tau$. Consequently, an ϵ change in $\hat{W}(2)$ would yields a perturbation for which no congruent steady cycle is representable. □

Lemma 24 *If a FHME is generically valid, then one of the following holds: (A) Overflow is strictly binding in period D^F of the FHME, and Equation 29 holds.*
(B)Current capacity is strictly binding in period D^F of the FHME, and Equation 30 holds.

$$\frac{\partial \pi_D^F}{\partial h_D^F} < r\left(\frac{\partial \pi_1^F}{\partial h_1^F} + \frac{\partial \pi_1^F}{\partial q_1^F}\cdot\frac{\partial Q_{t+1}^F}{\partial h_1^F}\cdot\mathcal{H}_1^+\right) \tag{29}$$

$$\frac{\partial \pi_D^F}{\partial h_D^F} > r\left(\frac{\partial \pi_1^F}{\partial h_1^F} + \frac{\partial \pi_1^F}{\partial q_1^F}\cdot\frac{\partial Q_1^F}{\partial h_1^F}\cdot\mathcal{H}_1^-\right) \tag{30}$$

Proof: Say that $d(\tau) = D$ and that the mimicked MPE has entered the steady cycle by period τ. By Lemma 3, Equation 29 (resp. Equation 30) holds if and only if the constraint in period $t=\tau$ binds strictly in the infinite horizon model.

We assume that neither of the suggested outcomes holds, and run through the possibilities for period $\gamma = D^F$. For each such possibility, we show that the FHME is not generically valid.

(1) Let us first say that overflow binds in $\gamma = D^F$, but we are not in Case (A) of the Lemma. If Equation 29 fails, then overflow must bind weakly in period $t=\tau$. From Lemma 22 we know that this means that the MPE is not generic, and so the FHME is not generically valid. If on the other hand, we are not in case (A) because overflow is weakly binding in period $\gamma = D^F$, then it would have to be strictly binding in period

$\gamma = D^F - 1$. Hence inflows in period $\gamma = D^F$ must equal exactly $\mathcal{C}_D^F$. A slight increase in w_D^F makes overflow bind strictly, so that the FHME is not generic.

(2) Say that current capacity binds in period $\gamma = D^F$, but Case (B) does not hold. If current capacity binds strictly in period $\gamma = D^f$, but Equation 30 fails, then current capacity is weakly binding in period $t = \tau$. If current capacity binds cleanly but weakly, then the steady cycle is not generic, so assume otherwise. From Lemma 17, this means that current capacity must be strictly binding in period $t = \tau - 2$ and that Equation 26 holds. Consider the block $\beta = \{d(\tau - 1), d(\tau)\}$. If we were to decrease $h_{d(\tau-1)}$ without decreasing $h_{d(\tau)}$, then we would violate the Euler Relation 16. Hence a decrease in $\hat{W}(\beta)$ must lead to a decrease in $h_{d(\tau)}$, which would result in a strictly binding constraint in period τ. That is, the MPE is not generic, which means that the FHME is not generically valid.

If, on the other hand, we are not in Case (B) but Equation 30 does hold, then current capacity must be weakly binding in period $\gamma = D^F$. Now Lemma 17 tells us that current capacity is strictly binding in period $\gamma = D^F - 2$ and that Equation 27 holds. Hence, if we decrease $w_{D-1}^F + w_D^F$ by a small amount, then current capacity must bind strictly in period $\gamma = D^F$, and the FHME is not generic.

The final way in which we might have case (B) fail despite a binding current capacity in period D^F is if current capacity binds weakly in period $\gamma = D^F$ and Equation 30 fails. Of course this means that current capacity binds weakly in period $t = \tau$ and $h_\tau = \mathcal{C}_\tau$. We first note that if current capacity bind cleanly in period $t = \tau$, then by Lemma 23 the MPE is not generic. If, instead, current capacity binds with kink in period $t = \tau$, then $\frac{\partial \pi_\tau}{\partial h_\tau} = r\left(\frac{\partial \pi_{\tau+1}}{\partial h_{\tau+1}} + \frac{\partial \pi_{\tau+1}}{\partial q_{\tau+1}} \cdot \frac{\partial Q_{\tau+1}}{\partial h_{\tau+1}} \cdot H_{\tau+1}^+\right)$. Let β denote the block starting on date $d(\tau + 1)$. We know that if we increase $\hat{W}(\beta)$, then $h_{\tau+1}$ must increase. This leads to a strictly binding current capacity constraint on τ. Hence the mimicked MPE is not generic, and so the FHME is not valid.

Finally we consider the case in which no constraint binds in period $\gamma = D^F$. In this case $h^F_D = \mathcal{C}^F_D$ and no constraint binds in period $t = \tau$. Let $d = d(\tau)$, and let β denote the block to which d belongs. Assume first that β ends with a cleanly binding constraint or else that d is in the first sub-block of β. In either of these cases, h_d is linearly increasing in $\hat{W}(\beta)$. On the other hand, if d is in the second sub-block of β, then we know that h_d is a linear increasing function of $\hat{W}(\beta')$ for other block β'. In either case, there is d such that if we added ϵ to w_d, then this would increase h_d. On the other hand, so long as ϵ is sufficiently small this would not change h^F_D. That is, the FHME is not generically valid.□

Proof of Proposition 13: If $\frac{dH_1^F}{dR_1^F}$ is defined, then Proposition 13 and Lemma 24 are the same. □

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