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The axisymmetric Prandtl–Batchelor eddy behind a circular disc in a uniform stream

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Analytical support is given to Fornberg's numerical evidence that the steady axially symmetric flow of a uniform stream past a bluff body has a wake eddy which tends towards a large Hill's spherical vortex as the Reynolds number tends to infinity. The viscous boundary layer around the eddy resembles that around a liquid drop rising in a liquid, especially if the body is a circular disc, so that the boundary layer on it does not separate. This makes it possible to show that if the first-order perturbation of the eddy shape from a sphere is small then the eddy diameter is of order $R^{1/5}$ times the disc diameter, where R is the Reynolds number based on the disc diameter. Previous authors had suggested $R^{1/3}$ and $\ln R$, but they appear to have made unjustified assumptions.

1. Introduction

Batchelor (1956*b*) said "The determination of the steady flow about a bluff body placed in a uniform stream of incompressible viscous fluid at large Reynolds number (where the word 'steady' in this context implies that somehow turbulence has been suppressed) is an old problem, for which no completely satisfactory solution is available". Although that was over 40 years ago, Batchelor's statement is still true for the axisymmetric case. One major advance since 1956 is the work of Fornberg (1988), who gave a numerical solution for the steady flow past a rigid sphere at Reynolds numbers R up to 5000. This solution lacks physical realism because the steady flow is unstable, but it is still interesting because it reveals a wake eddy much larger than the sphere, which resembles Hill's (1894) spherical vortex more and more closely as R increases. In Hill's vortex and in the interior of Fornberg's eddy the vorticity ω obeys the well-known condition (Prandtl 1905; Batchelor 1956*a*) that

$$\Omega = \omega/m = \text{constant}, \tag{1}$$

where m is (cylindrical polar) distance from the axis of symmetry. Around the eddy is a viscous boundary layer which divides at the rear stagnation point to go each way along the axis, downstream into the far wake and upstream back towards the body.

However, the asymptotic size of that eddy for large R is not well determined. Batchelor (1956*b*) suggested that (eddy radius)/(body radius) $= O(1)$ from physical considerations but without detailed calculation, Fornberg (1988) suggested $O(\ln R)$ as an empirical fit to his data, while Parlange (1969) suggested $O(R^{1/3})$, by calculating the drag due to viscous dissipation in the irrotational flow and the Hill's vortex, and equating it to the drag that would result from the Bernoulli pressure difference between front and rear of the body. On the assumption that flow inside the eddy can be neglected near the body, Parlange obtained the ratio of eddy radius to body radius as $(3R/320)^{1/3} = 0.2109R^{1/3}$. As the actual interior flow is nearly as fast as the exterior flow, the pressure drag is much less than Parlange's, and his method then shows that the ratio of radii must be much less than $0.2109R^{1/3}$. Chernyshenko (1995) also suggested a variation as $R^{1/3}$, by giving more detail of the boundary layers implicit in Parlange (1969), rediscovering various results of Harper & Moore (1968) in the process, and applying them to the present problem. He obtained his $O(R^{1/3})$ result by assuming that an unevaluated constant k_d in his theory was of order unity. Parlange's method suggests that $k_d \ll 1$; a major reason for undertaking the present work is to find how large the eddy is and how small k_d is.

The purposes of this paper are, first, to test whether Hill's spherical vortex can be the correct limit as $R \to \infty$, but in the easier case of a disc set across the stream instead of a sphere, by finding a first-order approximation to the eddy shape which holds everywhere except near the rear stagnation point, and secondly, to find how the size of that vortex varies with R.

It will transpire that the wake eddy is indeed nearly spherical, but with a diameter different from that suggested by any of the previous workers: $O(R^{1/5})$ times as large as that of the disc.

2. Mathematical formulation

Let a circular disc of radius a be at rest in a liquid of kinematic viscosity ν, density ρ and dynamic viscosity $\mu = \rho\nu$ which flows steadily past the disc in a direction perpendicular to its plane at speed U. Let the Reynolds number $R = 2Ua/\nu$. Let a' be the radius transverse to the stream of the dividing streamline Σ between the outer flow and the wake eddy (figure 1), let $\alpha = a/a'$, and let $R' = 2Ua'/\nu = R/\alpha$ be the Reynolds number based on the eddy size. The major problem to be addressed is how α varies with R when R is large. The discussion above of the work of Parlange (1969) shows that $\alpha \gg (320/3R)^{1/3} = 4.743R^{-1/3}$, and Fornberg (1988) showed that at least up to $R = 5000$ the shape of Σ behind a spherical body became closer to a sphere as R increased, with Ω defined by (1) constant inside it and away from boundary layers. If Σ were exactly a sphere of radius a', Hill's (1894) stream functions $\overline{\psi}_0$ outside and $\overline{\psi}_1$ inside Σ would be

$$\overline{\psi}_0 = \tfrac{1}{3}V_0\left(r^2 - \frac{a'^3}{r}\right)\sin^2\theta, \tag{2}$$

$$\overline{\psi}_1 = \tfrac{1}{2}V_1\left(\frac{r^4}{a'^2} - r^2\right)\sin^2\theta, \tag{3}$$

in terms of spherical polar coordinates (r, θ) with origin at the centre of Σ and with $\theta = 0$ pointing downstream. Figure 2 shows Hill's streamlines on one side of the axis of symmetry.

The speeds V_0, V_1 in (2), (3) are respectively the maximum speeds of the fluid outside and inside the vortex. They are reached at the equator and the centre; in Hill's first approximation $V_0 = V_1 = \frac{3}{2}U$, and $\Omega = 0$ outside Σ, $\Omega = 10V_1/a'^2 = 15U/a'^2$ inside Σ.

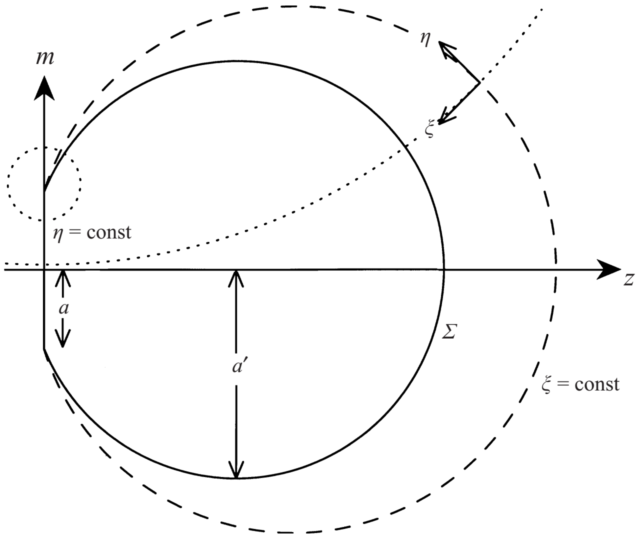

FIGURE 1. The disc of radius a and wake eddy Σ, approximately a sphere of radius a'. Toroidal coordinates ξ, η are constant on spherical caps (dashed) and tori (dotted) respectively. The arrows marked ξ and η indicate the directions in which those coordinates increase.

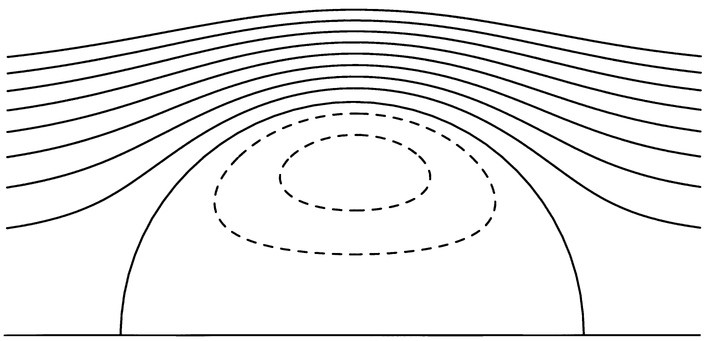

FIGURE 2. Streamlines of the first approximation to the flow: $\overline{\psi}_0$ (solid) outside the sphere, $\overline{\psi}_1$ (dashed) inside it.

3. Inviscid perturbations of shape

Assuming that $\alpha \ll 1$ and that the shape of Σ is close to a sphere of radius a', let us consider the inviscid theory for the irrotational flow outside Σ and Prandtl–Batchelor flow inside Σ. To do this, we first define cylindrical polar coordinates (m, z) centred on the disc, with $z > 0$ downstream of it, and toroidal coordinates (ξ, η) by

$$m = \frac{as}{q(\xi,\eta)^2} = r \sin\theta, \tag{4}$$

$$z = \frac{a \sin\xi}{q(\xi,\eta)^2} = r\cos\theta + (a'^2 - a^2)^{1/2}, \tag{5}$$

where we write for brevity throughout this paper

$$q(\xi,\eta) = (c - \cos\xi)^{1/2}, \quad c = \cosh\eta, \quad s = \sinh\eta, \quad q_0 = q(\xi_0,\eta);$$

$\xi = -\pi$ on the upstream side of the disc itself, $-\pi < \xi < 0$ in the whole region upstream of it, $\xi = \pi$ on the downstream side of the disc itself, $0 < \xi < \xi_0 = \sin^{-1}\alpha$

in the region downstream from the disc and outside the unperturbed sphere $r = a'$, and $\xi_0 < \xi < \pi$ inside that sphere. The coordinate surfaces of constant ξ are caps of spheres bounded by the disc, and the surfaces of constant η are tori orthogonal to those caps, with $\eta \to 0$ on the axis of symmetry, $\eta \to \infty$ on the edge of the disc. As $r \to \infty$ ('physical infinity') ξ and η both tend to zero; the first of the following identities gives an easy way to prove this, and the second is needed when finding $\overline{\psi}_0$:

$$\frac{m^2+z^2}{a'^2} = \frac{c+\cos\xi}{c-\cos\xi},$$
$$\frac{r}{a'} = \frac{q(2\xi_0-\xi,\eta)}{q(\xi,\eta)}.$$

In these toroidal coordinates the stream functions $\overline{\psi}_0, \overline{\psi}_1$ of (2), (3) are

$$\overline{\psi}_0 = \frac{V_0 a^2 s^2}{3q(\xi,\eta)^4}\left\{1 - \frac{q(\xi,\eta)^3}{q(2\xi_0-\xi,\eta)^3}\right\}, \tag{6}$$

$$\overline{\psi}_1 = \frac{V_1 a^2 \alpha s^2 \sin(\xi_0-\xi)}{q(\xi,\eta)^6}, \tag{7}$$

and the actual external and internal inviscid stream functions ψ_0, ψ_1 are irrotational perturbations of $\overline{\psi}_0$ and $\overline{\psi}_1$, where the latter has a suitable value of V_1 determined by boundary-layer analysis like that of Harper & Moore (1968), because the internal flow must still have constant Ω, though a slightly smaller constant than before.

Because the mapping $(m,z) \to (\xi,\eta)$ is conformal, an irrotational stream function ψ_I obeys

$$\frac{\partial}{\partial\xi}\left(\frac{1}{m}\frac{\partial\psi_I}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial\psi_I}{\partial\eta}\right) = 0; \tag{8}$$

the general solution of this equation with the variables separated may be written

$$\psi_I = \frac{s}{q}\{k_1\cos\lambda\xi + k_2\sin\lambda\xi\}\{k_3 P^1_{\lambda-1/2}(c) + k_4 Q^1_{\lambda-1/2}(c)\}, \tag{9}$$

where P, Q denote associated Legendre functions, and $k_1, \ldots, k_4$ and λ are arbitrary constants. Without loss of generality we may put

$$-\tfrac{1}{2}\pi < \arg\lambda \leqslant \tfrac{1}{2}\pi. \tag{10}$$

The velocity components v_ξ, v_η obey

$$v_\xi = -\frac{q^4}{a^2 s}\frac{\partial\psi}{\partial\eta}, \qquad v_\eta = \frac{q^4}{a^2 s}\frac{\partial\psi}{\partial\xi}, \tag{11}$$

whether the stream function ψ is rotational or irrotational.

As the velocity is finite on the symmetry axis $\eta = 0$, k_4 must be zero, and as the velocity is finite at the edge of the disc where $\eta \to \infty$, $\mathrm{Re}(\lambda) \leqslant 0$ by (A 6) in the Appendix, which with (10) makes $\lambda = \mathrm{i}t$ where $t \geqslant 0$, and so

$$\psi_0 = \overline{\psi}_0 + \frac{s}{q(\xi,\eta)}\mathscr{M}[C_0(t)\cosh(\pi+\xi)t - S_0(t)\sinh(\pi+\xi)t\,;\eta], \tag{12}$$

$$\psi_1 = \overline{\psi}_1 + \frac{s}{q(\xi,\eta)}\mathscr{M}[C_1(t)\cosh(\pi-\xi)t - S_1(t)\sinh(\pi-\xi)t\,;\eta], \tag{13}$$

where C_0, S_0, C_1, S_1 are four functions to be found from the boundary conditions, and the Mehler–Fock operator $\mathscr{M}$ is defined by

$$\mathscr{M}[f(t);\eta] = \int_0^\infty P^1_{\mathrm{i}t-1/2}(c)f(t)\,\mathrm{d}t = f_M(c), \text{ say,} \tag{14}$$

where P denotes a Legendre function of complex order. The inverse operator $\mathscr{M}^{-1}$ obeys (Luke 1969, p. 178)

$$\mathscr{M}^{-1}[f_M(c);t] = f(t) = \frac{t\tanh \pi t}{\frac{1}{4}+t^2}\int_1^\infty P^1_{\mathrm{i}t-1/2}(c)f_M(c)\,\mathrm{d}c. \tag{15}$$

Mehler–Fock transforms of various functions which occur in this paper have been collected in the Appendix, because they take some effort to extract from the usual reference books.

On $\xi = \pi$ we have $\psi_1 = 0$, on $\xi = -\pi$ we have $\psi_0 = 0$, and so (12) and (13) give

$$\mathscr{M}[C_j(t);\eta] = \frac{V_j a^2\alpha^2 s}{(c+1)^{5/2}}, \tag{16}$$

for $j = 0, 1$, because $\xi_0 \ll 1$; terms of order ξ_0^4 have been neglected. Hence

$$C_j(t) = -\frac{4\sqrt{2}\,V_j a^2\alpha^2}{3}\frac{t^2}{\cosh \pi t}. \tag{17}$$

The boundary conditions $\psi_0 = \psi_1 = 0$ on the eddy boundary Σ, which is given in terms of a function h by $\xi = \xi_0 + h(\eta)$, are to leading order in h and ξ_0,

$$\frac{V_j a^2\alpha h(\eta)s}{q_0^5} = \mathscr{M}[C_j(t)\cosh \pi t - S_j(t)\sinh \pi t\,;\eta], \quad j = 0,1, \tag{18}$$

where $(\pi \pm \xi_0)t$ has been approximated by πt because $\xi_0 \ll 1$. The wake dividing streamline must leave the disc tangentially, and so $h(\eta) \to -\xi_0 = -\alpha + O(\alpha^3)$ as $\eta \to \infty$. Elsewhere around Σ, consistency of the assumption that its shape is close to a sphere of radius a' requires that $|h(\eta)| \ll \alpha$, a condition which we now check.

Defining a function $d(t)$ by

$$\frac{h(\eta)s}{q_0^5} = \mathscr{M}[d(t);\eta], \tag{19}$$

we have

$$C_j(t)\cosh \pi t - S_j(t)\sinh \pi t = V_j a^2\alpha d(t), \tag{20}$$

and finding S_j and hence d will enable us to find h.

The remaining boundary condition comes from Bernoulli's theorem applied to the fluid just inside Σ and just outside Σ. The speed of the fluid is close to the η velocity component $v_{\eta j}$ given for $j = 0, 1$ by

$$v_{\eta j} = \frac{q(\xi,\eta)^4}{a^2 s}\frac{\partial \psi_j}{\partial \xi}, \tag{21}$$

so that

$$v_{\eta 0} = -V_0\alpha s\left(\frac{1}{q_0^2} - \frac{h\alpha}{q_0^4}\right) + \frac{q_0^3}{a^2}\mathscr{M}[tC_0(t)\sinh \pi t - tS_0(t)\cosh \pi t;\eta], \tag{22}$$

$$v_{\eta 1} = -V_0\alpha s\left(\frac{1}{q_0^2} + \frac{4h\alpha}{q_0^4}\right) - \frac{q_0^3}{a^2}\mathscr{M}[tC_1(t)\sinh \pi t - tS_1(t)\cosh \pi t;\eta], \tag{23}$$

where $(\pi \pm \xi)t$ has again been approximated by πt, but not before differentiating with respect to ξ. Thus the Bernoulli constant B, given by $B = v_{\eta 0}^2 - v_{\eta 1}^2$ evaluated on Σ, is

$$B = (V_0 + V_1)\frac{\alpha s}{q_0^2}\left\{(V_0 - V_1)\frac{\alpha s}{q_0^2} + (V_0 + 4V_1)\frac{\alpha^2 hs}{q_0^4} + \frac{q_0^3}{a^2}\mathscr{M}\left[t\{S_0(t) + S_1(t)\}\cosh \pi t - t\{C_0(t) + C_1(t)\}\sinh \pi t\,; \eta\right]\right\} \tag{24}$$

to leading order. Substitution for C_0, C_1, S_0, S_1, V_0–V_1 from (17), (20), (24) leads, if

$$b = \frac{V_0^2 - V_1^2}{B} = \frac{V_0^2 - V_1^2}{v_{\eta 0}^2 - v_{\eta 1}^2}, \text{ and } B_V = \frac{B}{(V_0 + V_1)^2} \sim \frac{B}{9U^2}, \tag{25}$$

where b and B_V are dimensionless, to

$$B_V\left\{\frac{1}{\alpha^2 q_0 s} - \frac{bs}{q_0^5}\right\} - \frac{(V_0 + 4V_1)\alpha hs}{(V_0 + V_1)q_0^7} = -\mathscr{M}\left[\frac{8\sqrt{2}\,\alpha t^3}{3\sinh 2\pi t} + \frac{t\,d(t)}{\tanh \pi t}\,; \eta\right]. \tag{26}$$

The right-hand side of (26) would be changed by $O(\alpha^2)$ of itself without the approximation made in (18), not merely $O(\alpha)$ as one might have supposed, because the corrections of that order cancel out. Fortunately, $O(\alpha^2)$ is small enough to ignore in what follows.

If we also ignore the term involving $\alpha hs/q_0^7$ on the left-hand side of (26), a simplification which is valid if $\eta \gg \alpha$, various equations in the Appendix give

$$-d(t) = B_V\sqrt{2}\sinh \pi t\left\{\frac{\sinh(\pi - \xi_0)t - \cos\frac{1}{2}\xi_0 \sinh \pi t}{\alpha^3(\frac{1}{4} + t^2)\cosh^2 \pi t} + \frac{4b\sinh(\pi - \xi_0)t}{3\alpha\cosh^2 \pi t}\right\} + \frac{4\sqrt{2}\,\alpha t^2}{3\cosh^2 \pi t}, \tag{27}$$

and hence

$$\frac{h(\eta)s}{q_0^5} = \frac{4B_V}{\pi\alpha^3 s}\left\{q_0\chi' - (c-1)^{1/2}\chi_0'\cos\tfrac{1}{2}\xi_0 - \frac{\pi\alpha}{2\sqrt{2}} + O(\alpha^2) + \frac{\alpha^2 s^2 b}{3}\left[\frac{\chi'}{q_0^3} + \frac{\sqrt{2}\cos\frac{1}{2}\xi_0}{(1+c)q_0^2}\right]\right\} - \frac{2\alpha s\chi_0}{\pi(c-1)^{5/2}} + \frac{2\sqrt{2}\,\alpha(c+2)}{3\pi(c-1)s}, \tag{28}$$

where

$$\chi = \tfrac{1}{2}\pi - \chi' = \tan^{-1}(q_0/\sqrt{2}\cos\tfrac{1}{2}\xi_0), \tag{29}$$

$$\chi_0 = \tfrac{1}{2}\pi - \chi_0' = \tan^{-1}(\sinh\tfrac{1}{2}\eta) = \lim_{\xi_0 \to 0}\chi. \tag{30}$$

It is now possible to evaluate the dimensionless Bernoulli constant B_V, because in the limit $\eta \to \infty$ we have $h(\eta) \to -\alpha$, $c \sim q_0^2 \sim s \to \infty$, $\chi_0' \sim (2/s)^{1/2}$, $\chi' \sim (\cos\frac{1}{2}\xi_0)(2/s)^{1/2}$, and (28) then gives

$$-\alpha s^{-3/2} = \frac{\sqrt{2}\,B_V}{\alpha^3 s}\left[-\alpha + \frac{2\sqrt{2}\alpha^2 b}{3} + O(s^{-1}) + O(\alpha^2)\right] - \alpha s^{-3/2} + \frac{2\sqrt{2}\,\alpha s^{-1}}{3\pi}. \tag{31}$$

The leading terms for large s and small α give

$$B_V \sim \frac{2\alpha^3}{3\pi - 8\alpha b}. \tag{32}$$

Physically, the eddy goes round slower than inviscid theory would suggest, and so $B_V > 0$, and $b < 3\pi/8\alpha$.

Hence, for $\eta \gg \alpha = o(1)$,

$$\frac{\pi s^2 h}{q_0^5} \sim \frac{8\alpha}{3\pi - 8\alpha b}\left\{\frac{-\pi}{2\sqrt{2}} + \frac{\alpha b s^2}{3q_0^2}\left[\frac{\chi_0'}{q_0} + \frac{\sqrt{2}}{(1+c)}\right]\right\} - \frac{2\alpha s^2\chi_0}{(c-1)^{5/2}} + \frac{2\sqrt{2}\alpha(c+2)}{3\pi(c-1)}. \quad (33)$$

For $\eta \gg 1$, (33) gives

$$h(\eta) \sim -\alpha + \frac{4\sqrt{2}\alpha}{\pi s^{1/2}}\left[1 + \frac{4\alpha b}{3(3\pi - 8\alpha b)}\right], \quad (34)$$

so that Σ leaves the edge of the disc with $z \propto (m-a)^{3/2}$, which implies the usual infinite curvature for a dividing free streamline where it leaves a body. For small (but not too small) η such that $\alpha \ll \eta \ll 1$,

$$s \sim \eta, \quad q_0 \sim \eta/\sqrt{2}, \quad \chi_0 \sim \tfrac{1}{2}\eta, \quad \chi \sim \tfrac{1}{2}\eta, \quad c \sim 1 + \tfrac{1}{2}\eta^2, \quad (35)$$

and (33) gives

$$h \sim -\frac{\alpha\eta^2(3\eta - 4\alpha b)}{6(3\pi - 8\alpha b)}. \quad (36)$$

However, as most of the dividing streamline Σ is in a region where $\eta = O(\alpha)$, it is necessary to examine h for such values of η in order to check whether $|h| \ll \alpha$ there, which is a necessary condition for consistency of the theory. That is the subject of §3.1.

3.1. *Small* η

For $\eta \ll 1$ the foregoing theory allows several simplifications. By (A 2) in the Appendix and Gradshteyn & Ryzhik (1980, equation 8.715.1),

$$\begin{aligned} P^1_{\mathrm{i}t-1/2}(\cosh\eta) &= -\frac{2^{1/2}(\frac{1}{4}+t^2)}{\pi^{1/2}\Gamma(\frac{3}{2})\sinh\eta}\int_0^\eta \cos(xt)(\cosh\eta - \cosh x)^{1/2}\,\mathrm{d}x \\ &\sim -\frac{2(\frac{1}{4}+t^2)}{\pi}\int_0^{\pi/2}\cos(\eta t\sin\vartheta)\,\eta\cos^2\vartheta\,\mathrm{d}\vartheta \qquad \text{for } \eta \ll 1, \end{aligned}$$

which, by Gradshteyn & Ryzhik (1980, equation 3.715.10),

$$= -\frac{(\frac{1}{4}+t^2)J_1(\eta t)}{t}, \quad (37)$$

and so if $\eta = H\alpha$, $t = T/\alpha$, $\alpha \ll 1$, $H = O(1)$, $f(t) = F(T)$, we have $s \sim \alpha H$, $q_0 \sim \alpha(H^2+1)^{1/2}/\sqrt{2}$, and

$$\begin{aligned} \mathscr{M}[f(t);\eta] &\sim -\frac{1}{\alpha^2}\int_0^\infty TF(T)J_1(HT)\,\mathrm{d}T \\ &= -\frac{1}{\alpha^2}\mathscr{H}_1[F(T);H], \end{aligned} \quad (38)$$

where $\mathscr{H}_1$ denotes a Hankel transform. If $T^{1/2}F(T)$ is piecewise continuous and absolutely integrable on the positive real line and $\mathscr{H}_1[F(T);H] = G(H)$, then $\mathscr{H}_1[G(H);T] = F(T)$ (Sneddon 1972).

If $W(T)$ is defined to be $\alpha^2 d(t)/4\sqrt{2}$, $X(H)$ to be $Hh(\eta)/(H^2+1)^{5/2}$, (19) becomes

$$X(H) = -\mathscr{H}_1[W(T);H], \quad (39)$$

and equations (26), (32) give

$$\frac{c_1}{H(H^2+1)^{1/2}}-\frac{c_2 H}{(H^2+1)^{5/2}}-\frac{VX(H)}{H^2+1}=\mathscr{H}_1\left[\frac{2T^3\alpha}{3\sinh(2\pi T/\alpha)}+\frac{TW(T)}{\tanh(\pi T/\alpha)};H\right], \quad (40)$$

where $c_1=\alpha^4/2(3\pi-8\alpha b)$, $c_2=4bc_1$, $V=(2V_0+8V_1)/(V_0+V_1)\sim 5-3Bb/(V_0+V_1)^2$, and (36) becomes

$$X(H)\sim -\frac{c_1}{H}+\frac{c_2}{3H^2} \text{ if } 1 \ll H \ll \alpha^{-1/2}. \quad (41)$$

The term $2T^3\alpha/3\sinh(2\pi T/\alpha)$ in (40) is readily shown, by expanding $J_1(TH)$ in its Taylor series about $H=0$, to give a leading-order contribution $\alpha^5 H/512$ to (40) if $|H| \ll \alpha^{-1}$, and so to be negligible in our region $H \ll \alpha^{-1/2}$. It is also a good approximation in (40) to replace $\tanh(\pi T/\alpha)$ by 1. With those approximations, (40) becomes

$$\frac{c_1}{H(H^2+1)^{1/2}}-\frac{c_2 H}{(H^2+1)^{5/2}}-\frac{VX(H)}{H^2+1}=\mathscr{H}_1[TW(T);H], \quad (42)$$

which we must now solve. If $c_1=0$, a brief search in Gradshteyn & Ryzhik (1980) reveals that one solution is

$$W(T)=\frac{c_2\,\mathrm{e}^{-T}}{\pi(V-3)}, \quad X(H)=\frac{-c_2 H}{(V-3)(H^2+1)^{3/2}}; \quad (43)$$

fortunately V is very close to 5, not 3. This solution is not unique, but the problem is linear: its general solution is a particular solution with $c_1=0$ plus the general solution with $c_2=0$.

If, then, we now put $c_2=0$, we exploit the identities

$$\int_0^\infty \frac{H^2 J_1(HT)}{(H^2+\sigma^2)^{5/2}}\mathrm{d}H=\frac{T\,\mathrm{e}^{-\sigma T}}{3\sigma}, \quad \int_0^\infty \frac{H^2 J_1(HT)}{(H^2+\sigma^2)^{3/2}}\mathrm{d}H=\mathrm{e}^{-\sigma T}, \quad (44)$$

by multiplying each side of (42) by $H^2(H^2+1)/(H^2+\sigma^2)^{5/2}$ and integrating with respect to H from 0 to ∞. This converts the Hankel transforms into Laplace transforms and the integral equation (42) into a first-order linear differential equation:

$$c_1\frac{1+\sigma+\sigma^2}{\sigma^2(1+\sigma)}=\int_0^\infty T^2W(T)\,\mathrm{e}^{-\sigma T}\{3\sigma-V-(\sigma^2-1)T\}\,\mathrm{d}t. \quad (45)$$

If $L(\sigma)=\int_0^\infty T^2W(T)\,\mathrm{e}^{-\sigma t}\,\mathrm{d}t$, and we use the approximation $V=5$, (45) gives

$$(\sigma^2-1)\frac{\mathrm{d}L(\sigma)}{\mathrm{d}\sigma}+(3\sigma-5)L(\sigma)=c_1\frac{1+\sigma+\sigma^2}{\sigma^2(1+\sigma)}, \quad (46)$$

of which the solution is

$$L(\sigma)=c_1\left[\frac{1}{\sigma}-\frac{1}{\sigma+1}+\frac{5\ln\sigma-10}{(\sigma+1)^3}-\frac{10\ln\sigma}{(\sigma+1)^4}\right]+c_3\left[\frac{1}{(\sigma+1)^3}-\frac{2}{(\sigma+1)^4}\right], \quad (47)$$

where c_3 is an arbitrary constant.

The complete solution to (42) for any c_1 and c_2 is thus

$$T^2W(T)=c_1\left[1+\mathrm{e}^{-T}\{-1-5T^2+5f_3(T)-10f_4(T)\}\right] +\tfrac{1}{2}c_2T^2\,\mathrm{e}^{-T}+c_3\,\mathrm{e}^{-T}\{\tfrac{1}{2}T^2-\tfrac{1}{3}T^3\}, \quad (48)$$

if we put

$$f_n(T) = -\frac{T^n}{n!}\,{}_2F_2(1,1;2,n+1;T) + \frac{T^{n-1}}{(n-1)!}\{\psi(n) - \ln T\}, \tag{49}$$

where ${}_2F_2$ is the generalized hypergeometric function and $\psi(n)$ is the logarithmic derivative of the gamma function (in its standard notation, which is needed in this paragraph only; elsewhere in this paper of course ψ denotes a stream function). The Hankel transform of $W(T)$ converges, and it turns out that

$$X(H) \sim -c_1\left[\frac{1}{H} + \frac{-11+5\psi(3)}{2H^2}\right] - \frac{c_2}{2H^2} - \frac{c_3}{2H^2}, \tag{50}$$

for large H, which agrees with (41) if

$$c_3 = \{11 - 5\psi(3)\}c_1 - \tfrac{5}{3}c_2 = c_1\{11 - 5\psi(3) - \tfrac{20}{3}b\} = c_1(6.386 - 6.667b) \tag{51}$$

to four figures.

Equation (42) thus has a unique admissible solution, which depends on α, B_V and b, and which is $O(\alpha^4)+O(\alpha^4 b) = O(\alpha^3)$. The shape of the wake eddy is indeed close to a sphere if $\alpha \ll 1$, except close to the rear stagnation point where the approximation leading to (21) does not hold and where one would expect on physical grounds a small spike pointing downstream, because the speed of the flow just outside Σ at its rear end is $B^{1/2}$. The spike should thus be of linear size $O(a' B_V^{1/2}) = O(a\alpha^{-1}B_V^{-1/2})$. It may not actually exist. There is no sign of it in Fornberg (1988), and it is possible that the detailed mechanics of the stagnation region at the rear of Σ does not require it. Harper & Moore (1968) investigated some of the properties of such stagnation regions, but they could appeal to surface tension to prevent a spike from existing. Moffatt & Moore (1978) did find a time-dependent spike growing from a suitably perturbed Hill's spherical vortex in inviscid fluid, but the present problem concerns steady flows in viscous fluid. The question clearly requires further work.

3.2. *Inviscid drag*

Because the only solid body in our problem is a thin circular disc set across the stream, viscous shear stress on it does not contribute directly to the drag force F, and we can estimate F and hence $C_D = F/(\frac{1}{2}\pi a^2 \rho U^2)$ by integrating the pressure difference between the two sides of the disc, which is $\Delta p = \frac{1}{2}\rho(B + v_1^2 - v_0^2)$. As the upstream and downstream sides are $\xi = -\pi$, $\xi = +\pi$,

$$F = \int_0^a 2\pi m \Delta p \,\mathrm{d}m = \pi a^2 \rho \int_1^\infty \frac{B + v_1^2 - v_0^2}{(c+1)^2}\,\mathrm{d}c. \tag{52}$$

Now

$$v_j = -\frac{V_j \alpha s}{c+1} + (-1)^i \frac{(c+1)^{3/2}}{a^2}\mathscr{M}[tS_j(t);\eta] \tag{53}$$

so that to leading order for small α

$$v_1 + v_0 = -(V_1 + V_0)\frac{\alpha s}{c+1}, \tag{54}$$

$$v_1 - v_0 = -(V_1 - V_0)\frac{\alpha s}{c+1} - (V_1 + V_0)(c+1)^{3/2}\mathscr{M}\left[\frac{4\sqrt{2}\alpha^2 t^3}{\sinh \pi t} + \frac{\alpha t d(t)}{\sinh \pi t};\eta\right] \tag{55}$$

and from (27), with the aid of the Appendix and Gradshteyn & Ryzhik (1980),

$$\frac{F}{\pi a^2 \rho} = B\left(\frac{1 - 2\ln 2 + 8\boldsymbol{G}/\pi}{2\sqrt{2}} + \frac{1}{2} - \frac{8\sqrt{2}\alpha b}{9}\right), \tag{56}$$

or

$$C_D = B_V(21.39 - 22.63\alpha b), \tag{57}$$

where $\boldsymbol{G}$ is Catalan's constant 0.915 965 594.... Numerical results such as (57) are given to four figures, though the original calculations were done with more.

4. Viscous theory

Section 3 shows that our model of the flow is consistent with a nearly spherical eddy if $R = \infty$, in which case equation (32) gives

$$\frac{v_0^2 - v_1^2}{(V_0 + V_1)^2} = B_V = \frac{2\alpha^3}{3\pi - 8\alpha b}. \tag{58}$$

If the boundary layer around Σ at large finite R is governed mainly by viscous forces around the major part of Σ rather than close to the disc, we may adapt the theory which Harper & Moore (1968) gave for a spherical drop. As we have the same fluid inside and outside Σ, and Σ is very close to a sphere of radius a', the viscous boundary layer around Σ smooths out discontinuities across Σ of $\frac{15}{2}\rho\nu U a'^{-1}\sin\theta$ in shear stress, and of $v_0 - v_1 = B_V(V_0 + V_1)^2/(v_0 + v_1) = 3B_V U \csc\theta$ in tangential velocity. Harper & Moore used a function $g(z)$, which is the dimensionless perturbation tangential velocity divided by $\sin\theta$ in a stagnation region, and which depends there only on z, their dimensionless scaled stream function. In the present problem it obeys the integral equation

$$\pi^{-1/2}\int_0^\infty g(z')\,\mathrm{e}^{-(z-z')^2}\,\mathrm{d}z' - g(z) = 5\sqrt{2}\,\mathrm{ierfc}(z) - 2\frac{B_V}{\delta_1}\,\mathrm{erfc}(z) \tag{59}$$

in Harper & Moore's notation, where $\delta_1^2 = \nu/a'U = 2/R'$. The solution is

$$g(z) = g_b(z) + 4B_V/\delta_1, \tag{60}$$

where g_b is given in table 1 of Harper & Moore (1968). As they explained, the boundary layer must merge into the flow outside it for large z, and so $g(\infty) = 0$. Because $g_b(\infty) = -5.826$ to four figures,

$$B_V = 2.060R'^{-1/2} = 2.060R^{-1/2}\alpha^{1/2}. \tag{61}$$

The drag coefficient C_D' based on the size of the eddy Σ is $120/R'$ to a first approximation, so that C_D based on the size of the disc is

$$C_D = \frac{120}{\alpha R} = 28.28B_V^2\alpha^{-2}, \tag{62}$$

by (61).

Equations (58), (57) and (62) imply that

$$(21.39 - 22.63\alpha b)(3\pi - 8\alpha b) = 56.57\alpha \ll 1. \tag{63}$$

Because $3\pi - 8\alpha b > 0$, αb must be $21.39/22.63 + O(\alpha)$, and hence to leading order

$$R = 3.685\alpha^{-5}, \tag{64}$$

$$\alpha = 1.298R^{-1/5}, \tag{65}$$

$$a'/a = \alpha^{-1} = 0.7704R^{1/5}, \tag{66}$$

$$B_V = 1.073\alpha^3 = 1.255R^{-3/5}, \tag{67}$$

$$C_D = 32.57\alpha^{-4} = 92.45R^{-4/5}, \tag{68}$$

$$v_{\eta 0}^2 - v_{\eta 1}^2 = 9.657\alpha^3 U^2, \tag{69}$$

$$V_0^2 - V_1^2 = 8.506\alpha^2 U^2. \tag{70}$$

Unfortunately, although (65) gives $\alpha^{-1} = 0.7704R^{1/5}$ which is indeed much smaller for large R than Parlange's (1969) value of $\alpha^{-1} = 0.2109R^{1/3}$, as it has to be, it does not actually become smaller until $R > 16\,620$, and our theory cannot be expected to be good numerically until R is well over 10^5. Hence detailed numerical comparisons with Fornberg (1988) are not worth doing: he stopped at $R = 5000$. The present theory would have to include various terms neglected because they were $O(\alpha)$ times those kept, and possibly even higher-order theory would be needed.

One can, however, check the approximations for consistency. We have ignored the viscous boundary layer on the disc which is, to a first approximation, the well-known similarity solution for an axisymmetric stagnation-point flow on a plane wall (see for example Rosenhead 1963, p. 419). Fortunately, we may. The contribution to C_D from the viscous dissipation in that layer is readily estimated as $O(\alpha^{-9/2}R^{-1/2}) = O(\alpha^7)$ which is much smaller than is given in (62). So is the contribution from the wake-like boundary layer around Σ due to the disc. That boundary layer can be analysed by using Mangler's transformation to reduce it to a two-dimensional flow of a non-uniform stream past a finite flat plate. Finally, the correction to B_V from inserting the downstream asymptotic form of that layer on the right-hand side of (59) is negligible.

5. Conclusions

As the wake eddy is of order $R^{1/5}$ times the size of the disc, the constant k_d which Chernyshenko (1995) assumed to be of order unity is actually $O(R^{-1/5})$. The $R^{1/5}$ power law contrasts intriguingly with the $O(R)$ behaviour found by Sychev (1967), Taganov (1968, 1970), Smith (1985), Peregrine (1985) and Chernyshenko (1988) in the two-dimensional analogue of the present problem. The axisymmetric calculation is simpler than theirs because Hill's spherical vortex has a simpler shape than Sadovskii's (1971) eddy obeying the two-dimensional Prandtl–Batchelor condition ω=constant. Both flows obey Batchelor's (1956*b*) conjectures for the vorticity distribution in the eddy, but neither obeys his conjecture that the size of the eddy is of the same order as the size of the body.

The precise nature of the flow near the downstream end of the eddy has not yet been elucidated. As a result, it has not yet been proved that the eddy really is close to a Hill's spherical vortex of radius $0.7704R^{1/5}$ times that of the disc. However, Hill's vortex is the only simply-connected axisymmetric shape known to obey the Prandtl–Batchelor condition in a uniform stream, and no evidence has been found in this paper that a small disc at the upstream end of the vortex will cause anything other than a small perturbation to the flow, except possibly at the rear end of Σ which may be a 'spike' of linear size $O(a'B_V^{1/2}) = O(a'R'^{-1/4}) = O(a\alpha^{1/2})$. Because that is much smaller than the size of Harper & Moore's (1968) effectively inviscid rear stagnation region, which is $O(a'R'^{-1/6})$, and as their method uses Ω as a function of ψ without needing it in terms of physical coordinates, it seems reasonable to conjecture that the present results would be affected only in higher approximations by the details of the spike. In any case, it may not be there: Fornberg (1988) did not find it.

The main results of the present work are that if the shape is close to Hill's vortex then the size must be $O(R^{1/5})$, and that there is no obvious reason why the shape should not be close to a sphere except in a small stagnation region.

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Appendix

Some properties of Mehler–Fock transforms and the Legendre functions arising in them take considerable effort to extract even from Sneddon (1972), who gives methods for evaluating the integrals, or Prudnikov, Brychkov & Marichev (1990), who give more of the relevant formulae than any other source known to the author. In the process of checking them, Abramowitz & Stegun (1972, section (8.13)), was found to be misleading. The complete elliptic integrals there are all $E(k)$ or $K(k)$, not the $E(k^2)$ or $K(k^2)$ which one would have expected in view of Abramowitz & Stegun, pp. 590, 591. Equation (A 3) below uses $E(k)$ and $K(k)$. Also, two errors were found in Prudnikov *et al.* (1990). Their equation 2.17.24.9 offers a finite value for $\int_0^\infty (\cos bx/\sinh^2 \pi x) P_{\mathrm{i}x-1/2}(c)\,\mathrm{d}x$, which actually diverges at $x = 0$, and their equation 2.17.24.6 is wrong. A correct version deducible from their equation 2.17.27.5 is

$$\int_0^\infty \frac{\cosh bx}{\cosh^2 \pi x} P_{\mathrm{i}x-1/2}(c)\,\mathrm{d}x = \frac{2^{1/2}}{\pi(c-\cos b)^{1/2}} \tan^{-1}\frac{(c-\cos b)^{1/2}}{(1+\cos b)^{1/2}} \quad \text{if } 0 \leqslant b < \pi \text{ and } c > 1. \tag{A 1}$$

Some of the equations below need the following identities as well as a Sneddon or Prudnikov equation to prove them; the asymptotic form (A 4) for $t \to \infty$ is from Robin (1959, p. 156). It can be used to prove that all the integrals in table 1 converge.

$$P^1_{\mathrm{i}t-1/2}(c) = -\left(\tfrac{1}{4}+t^2\right) P^{-1}_{\mathrm{i}t-1/2}(c) = \frac{\mathrm{d}}{\mathrm{d}\eta} P_{\mathrm{i}t-1/2}(c) \tag{A 2}$$

$$= \frac{1}{\sinh \frac{1}{2}\eta}\{E(\tanh \tfrac{1}{2}\eta) - K(\tanh \tfrac{1}{2}\eta)\} \quad \text{if} \quad t = 0 \tag{A 3}$$

$$\sim (2t/\pi s)^{1/2}\{\cos(t\eta + \tfrac{1}{4}\pi) + O(c/st)\} \quad \text{as} \quad t \to \infty, \tag{A 4}$$

$$\sim -\tfrac{1}{2}\left(\tfrac{1}{4}+t^2\right)\eta \quad \text{as} \quad \eta \to 0 \tag{A 5}$$

$$\sim \left(\frac{2}{\pi s}\right)^{1/2} \mathrm{Re}\left\{\frac{\Gamma(\mathrm{i}t)\mathrm{e}^{\mathrm{i}t\eta}}{\Gamma(\mathrm{i}t-\frac{1}{2})}[1 + O(s^{-2}) + O(ts^{-2})]\right\} \quad \text{as} \quad \eta \to \infty; \tag{A 6}$$

$$\frac{\sinh \pi t \sinh(\pi-\xi)t}{\cosh^2 \pi t} = \frac{\cosh(\pi-\xi)t}{\cosh \pi t} - \frac{\cosh \xi t}{\cosh^2 \pi t}. \tag{A 7}$$

The Mehler–Fock transforms in table 1 are valid if $0 \leqslant \xi < \pi$, $c = \cosh \eta > 1$, except for (A) and (A) whose integrals diverge if $\xi = 0$.

$f(t)$	$\mathscr{M}[f(t);\eta] = \int_0^\infty P^1_{\mathrm{i}t-\frac{1}{2}}(c) f(t)\,\mathrm{d}t$
$\dfrac{1}{\cosh \pi t}$	$-\dfrac{s}{2\sqrt{2}(1+c)^{3/2}}$
$\dfrac{t^2}{\cosh \pi t}$	$\dfrac{-3s}{4\sqrt{2}(1+c)^{5/2}}$
$\dfrac{t^3}{\sinh 2\pi t}$	$-\dfrac{3\eta}{1024} + O(\eta^3)$ if $\eta \ll 1$
$\dfrac{t^2}{\cosh^2 \pi t}$	$\dfrac{s}{\pi}\left\{\dfrac{3\tan^{-1}(\sinh\frac{1}{2}\eta)}{2\sqrt{2}(c-1)^{5/2}} - \dfrac{c+2}{2(c-1)s^2}\right\}$
$\dfrac{\tanh^2 \pi t}{(\frac{1}{4}+t^2)}$	$\dfrac{-2}{\pi s}\left\{2\sinh\tfrac{1}{2}\eta\cot^{-1}(\sinh\tfrac{1}{2}\eta) + \ln\dfrac{1+c}{2}\right\}$
$\dfrac{t\sinh(\pi-\xi)t}{\cosh \pi t}$	$\dfrac{-3s\sin\xi}{4\sqrt{2}\,q^5}$
$\dfrac{\sinh \pi t \sinh(\pi-\xi)t}{\cosh^2 \pi t}$	$\dfrac{-s}{\pi\sqrt{2}}\left\{\dfrac{1}{q^3}\tan^{-1}\dfrac{\sqrt{2}\cos\frac{1}{2}\xi}{q} + \dfrac{\sqrt{2}\cos\frac{1}{2}\xi}{(1+c)q^2}\right\}$
$\dfrac{\cosh(\pi-\xi)t}{\left(\frac{1}{4}+t^2\right)\cosh \pi t}$	$\dfrac{2\sin\frac{1}{2}\xi - \sqrt{2}\,q}{s}$
$\dfrac{t\sinh(\pi-\xi)t}{\left(\frac{1}{4}+t^2\right)\cosh \pi t}$	$\dfrac{\sin\xi}{\sqrt{2}\,sq} - \dfrac{\cos\frac{1}{2}\xi}{s}$
$\dfrac{\cosh \xi t}{\left(\frac{1}{4}+t^2\right)\cosh^2 \pi t}$	$\dfrac{-2}{\pi s}\left\{\sqrt{2}\,q\tan^{-1}\dfrac{q}{\sqrt{2}\cos\frac{1}{2}\xi} - \xi\sin\tfrac{1}{2}\xi - \cos\tfrac{1}{2}\xi\ln\dfrac{1+c}{2}\right\}$

TABLE 1. Mehler–Fock transforms used in the text; $c = \cosh\eta$, $s = \sinh\eta$, $q = (c - \cos\xi)^{1/2}$.

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