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Reducing Parabolic Partial Differential Equations to Canonical Form

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A simple method of reducing a parabolic partial differential equation to canonical form if it has only one term involving second derivatives is the following: find the general solution of the first-order equation obtained by ignoring that term and then seek a solution of the original equation which is a function of one more independent variable. Special cases of the method have been given before, but are not well known. Applications occur in fluid mechanics and the theory of finance, where the Black-Scholes equation yields to the method, and where the variable corresponding to time appears to run backwards, but there is an information-theoretic reason why it should.

1. Introduction

Since Black & Scholes (1973) wrote their influential paper on the valuation of options, linear parabolic partial differential equations of a kind long familiar in fluid mechanics have been appearing in the theory of finance also. A simple method for reducing such equations to the classical diffusion equation has been independently discovered three times, but is still not widely known. It deserves to be, because it is often simpler than other ways of achieving the same result. This paper gives some more special cases not covered in the previous work; no general theory yet describes in what circumstances the method is useful. Harper (1963) gave the special case $n = 1$, $a = 1$, d^0/x^0 and d^1 independent of x^0, in the notation of (1) below, and Srubshchik & Yudovich (1971) and Ruckenstein (1971) had the additional restriction $b = 0$. Srubshchik & Yudovich's method was extended by Batishchev (1975) to a coupled pair of equations.

Consider a linear equation of the form

$$w_{x^0x^0} + aw_t + bw + \sum_{i=0}^{i=n} d^i w_{x^i} = 0, \tag{1}$$

where subscripts indicate partial differentiation, superscripts indicate the various space-like coordinates (in this Introduction only), t is the time, and a, b, d^i are functions of x^j, t.

The method for reducing (1) to the diffusion equation is simply described:
(a) ignore the second derivative term for the moment,
(b) find the general solution of the resulting Lagrangian linear equation (Piaggio 1928 pp. 147–151) in the form $w = \Psi f(X^0, \ldots, X^{n-1}, T)$, where f is an arbitrary function of $n + 1$ variables, and Ψ, X^i, T are functions of x^j, t which are determined by that solution,
(c) seek a solution of the full equation (1) of the form $w = \Psi F(X^0, \ldots, X^n, T)$, where F is now a function of $n + 2$ variables, and X^n is independent of x^0 and is chosen to transform (1) into something as simple as possible. That may be the diffusion equation, or it may be an equation convertible into the diffusion equation by using the method again. There

are cases, such as a diffusion boundary layer on a rigid wall, where the method will reach $\partial f/\partial X = Y^\alpha \partial^2 f/\partial Y^2$, where α is a nonzero constant, instead of the ordinary diffusion equation $\partial f/\partial X = \partial^2 f/\partial Y^2$. For most purposes this is just as good: in particular, finding similarity solutions is just as easy, and so is the construction of Green's functions with simple boundary conditions. Levich (1962 p. 80–87) and Harper (1972 p. 119–120) did this with $\alpha = -1$. It should also be mentioned that seeking a similarity solution (Levich 1962, Batishchev & Srubshchik 1971) is a method for solving equations like (1) which is an alternative to that described herein.

We illustrate with three examples, one from each of Harper (1963), Hodgson *et al.* (1992) and Black & Scholes (1973); the last requires either two reductions of the present kind or a preliminary (fairly obvious) transformation of one of the independent variables.

2. A fluid mechanical example

If an otherwise uniform stream of viscous incompressible fluid past a body is started suddenly from rest and is then kept at constant speed at a Reynolds number $R \gg 1$, the flow is two-dimensional, and any instabilities have not yet had time to distort the laminar flow significantly, then the stream function ψ in the thin viscous boundary layer around the surface of the body (Goldstein 1965, p.118) obeys

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} = UU_x + \nu \psi_{yyy}, \tag{2}$$

where x is distance measured along the body, y is perpendicular distance from the body, t is the time, ν is the kinematic viscosity, and $U(x)$ is the speed of the flow at position x just outside the boundary layer. The equation is of course non-linear but if we consider only the outermost parts of the boundary layer we may write $\psi = U(y+\delta) + \epsilon$, where δ is the necessary modification of the displacement thickness in a time-dependent flow, and ϵ may be assumed to be very small if $y/\sqrt{\nu t}$ is large. Harper (1963) had neglected δ, Prof. K. Stewartson (personal communication 1966) pointed out the need for it, and Van Dommelen & Shen (1985) gave the full theory. A good first approximation to (2) for large $y/\sqrt{\nu t}$ is

$$\mathcal{L}(\zeta) \equiv \zeta_t + U\zeta_x - (U_x y + U_x \delta + U\delta_x)\zeta_y + U_x \zeta = \nu \zeta_{yy}, \tag{3}$$

where $\mathcal{L}$ is a first-order linear operator, and $\zeta = \epsilon_y$. We reduce (3) by the method outlined above. The equation $\mathcal{L}(\zeta) = 0$ has the subsidiary equations (Piaggio 1928, p.147)

$$\frac{dt}{1} = \frac{dx}{U} = -\frac{dy}{U_x y + U_x \delta + U\delta_x} = -\frac{d\zeta}{U_x \zeta}. \tag{4}$$

Three independent integrals of (4) are easily found to be

$$\begin{array}{rcccl} f &=& \zeta U &=& c_1, \\ T &=& t - \int U^{-1} dx &=& c_2, \\ Y &=& U(y+\delta) &=& c_3, \end{array}$$

where the c_i are arbitrary constants, and so the general solution of $\mathcal{L}(\zeta) = 0$ is

$$\zeta = U^{-1} f(Y, T),$$

where f is an arbitrary function. We now seek a solution of the full equation (3) of the form

$$\zeta = U^{-1} F(X, Y, T), \tag{5}$$

where X depends on x only. Direct substitution in (3) gives $F_X X_x = \nu U F_{YY}$ which reduces to the classical diffusion equation

$$F_X = F_{ZZ}, \tag{6}$$

if $X = \int U\, dx$, which is the velocity potential in the flow just beyond the boundary layer, and $Z = \nu^{-1/2} Y$, which is a boundary-layer approximation to the stream function (multiplied by the scaling constant $\nu^{-1/2}$).

Further developments in this problem need not be given here; see Van Dommelen and Shen (1985).

3. Okunev's equation

Recently Prof. J. Okunev encountered the following differential equation in a financial mathematical context (see Hodgson *et al.*, 1992) :

$$\tfrac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial A^2} + (\alpha e^{kt} - \beta A)\frac{\partial P}{\partial A} + \frac{\partial P}{\partial t} - \mu P = 0, \tag{7}$$

where P is a function of A and t, and $\sigma, \alpha, k, \beta, \mu$ are constants. As before, one starts by ignoring the second derivative term in (7), which amounts to pretending for the moment that $\sigma = 0$. The resulting Lagrangian linear equation has the subsidiary equations

$$\frac{dA}{(\alpha e^{kt} - \beta A)} = \frac{dt}{1} = \frac{dP}{\mu P}. \tag{8}$$

If $k + \beta \neq 0$ two independent solutions are easily found to be

$$\begin{array}{rcccl} f & = & Pe^{-\mu t} & = & c_1, \\ X & = & Ae^{\beta t} - \alpha e^{(k+\beta)t}/(k+\beta) & = & c_2, \end{array}$$

where c_1, c_2 are constants. If $k + \beta = 0$, X in (9) is replaced by $Ae^{\beta t} - \alpha t$, which is its limit as $k + \beta \to 0$ apart from a constant. Piaggio (1928) again gives the general solution of (8) as $c_1 = f(X)$, or $P = e^{\mu t} f(X)$, and we again seek a solution of the original equation (7) of the form

$$P = e^{\mu t} F(X, T), \tag{9}$$

where T is a function of t only, which now needs to be found.

Whether $k + \beta = 0$ or not, substitution from (9) into (7) now gives

$$\tfrac{1}{2}\sigma^2 e^{2\beta t} \frac{\partial^2 F}{\partial X^2} + \frac{\partial F}{\partial T}\frac{dT}{dt} = 0. \tag{10}$$

This is indeed the classical diffusion equation $F_{XX} = F_T$ if one puts $dT/dt = -\frac{1}{2}\sigma^2 e^{2\beta t}$, or $T = -\frac{1}{2}\sigma^2 t$ if $\beta = 0$, $T = -\sigma^2 e^{2\beta t}/4\beta$ if $\beta \neq 0$. Note that $dT/dt < 0$ if $\beta \geq 0$; the same feature occurs in the next example, and will be discussed there, where its meaning is more obvious.

4. The Black-Scholes formula

Black & Scholes (1973) showed that the value w at time t of an option to buy a stock worth x obeys

$$\tfrac{1}{2} v^2 x^2 w_{xx} + r x w_x + w_t - r w = 0, \tag{11}$$

where v^2 is the variance rate of return on the stock (assumed constant), and r is the short-term interest rate of a safe investment (also assumed constant). For "European" options which must be exercised either for a price c at a particular future time t^* or not at all, the appropriate boundary condition for (11) is

$$w(x, t^*) = (x - c)H(x - c), \tag{12}$$

where $H(z)$ is the Heaviside step function equal to 1 if $z \geq 0$, 0 if $z < 0$.

To reduce (11) to a form like (6), we begin by ignoring the w_{xx} term and obtaining

$$rxw_x + w_t = rw, \tag{13}$$

the subsidiary equations of which are

$$\frac{dx}{rx} = \frac{dt}{1} = \frac{dw}{rw}. \tag{14}$$

Two independent integrals of (14) are

$$\begin{array}{rcccl} f & = & we^{-rt'} & = & c_1, \\ S & = & \ln(x/c) - rt' & = & c_2, \end{array}$$

where c_1, c_2 are arbitrary constants, and we put $t' = t - t^*$ to simplify fitting the boundary conditions. The general solution of (13) is thus $c_1 = f(c_2)$, where f is an arbitrary function, i.e. $w = e^{rt'} f(S)$. We now seek the general solution of (11) in the form $w = e^{rt'} F(S, V)$, where S is defined in (15), and V is a function of t' which we find by substituting in (11). That gives

$$\tfrac{1}{2}v^2 F_{SS} - \tfrac{1}{2}v^2 F_S + F_V V_{t'} = 0, \tag{15}$$

which is not yet the diffusion equation but is simpler than (11). If we put $V = \frac{1}{2}v^2 t'$, so that (15) has the simplest possible coefficients, and then repeat the reduction process, we obtain

$$G_{XX} = G_T, \tag{16}$$

if $F(S, V) = G(X, T)$, where

$$\left.\begin{array}{rcccl} X & = & S + V & = & \ln(x/c) - (r - \frac{1}{2}v^2)t', \\ T & = & -V & = & -\frac{1}{2}v^2 t'. \end{array}\right\} \tag{17}$$

Alternatively, one might note the presence of $x^2 w_{xx}$ and xw_x in (11) and rewrite that equation in terms of $s = \ln(x/c)$ by analogy with the usual method for homogeneous linear ordinary differential equations (Piaggio 1928, pp.41–42). Only one pass through the reduction process is then needed, and the result is still (17), which is of course equivalent to the transformations of Black & Scholes (1973) though slightly simpler.

The boundary condition (12) becomes

$$G(X, 0) = c(e^X - 1)H(e^X - 1) = c(e^X - 1)H(X), \tag{18}$$

and the solution is now well-known (Carslaw & Jaeger 1959 pp.53, 71), being

$$\begin{aligned} G(X, T) &= \frac{c}{\sqrt{\pi}} \int_0^\infty (e^\xi - 1) e^{-(X-\xi)^2/4T}\, d\xi \\ &= \frac{c}{2}\left\{ e^{X+T} \operatorname{erfc}\left(-\frac{X}{2\sqrt{T}} - \sqrt{T}\right) - \operatorname{erfc}\left(-\frac{X}{2\sqrt{T}}\right)\right\}. \end{aligned} \tag{19}$$

As the cumulative normal distribution function $N(d) = \frac{1}{2}\text{erfc}(-d/\sqrt{2})$, our result (19) is equivalent to equation 13 of Black & Scholes (1973). It may also be derived from Gradshteyn & Ryzhik (1980) equation (3.322.1), which is

$$\int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[1 - \Phi\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right)\right] \tag{20}$$

in their notation, but they required that $\text{Re}\,\beta > 0$ and $u > 0$; (20) is actually valid for all u, positive, negative or complex, if $x \to \infty$ along the positive real axis.

Because $dT/dt < 0$, the time variable T in (17) runs backwards. This is presumably due to the direction in which information flows. In the physical problems where diffusion theory originated, entropy continually increases in a system left to itself, and information is gradually lost, but in Black & Scholes' world of investors continually hedging their portfolios, information is constantly arriving and being used.

5. Conclusions

If a linear parabolic partial differential equation has only one term involving second derivatives, it is worth trying the reduction described in this paper, possibly more than once, to transform it into the classical diffusion equation, because of the great variety of known methods for solving that. The reduction proceeds by (temporarily) ignoring the second derivative term, solving what remains, and then substituting an arbitrary function with one more independent variable into the full equation. Examples from two totally different fields of application show that the method can give useful results very simply. In the process, equation (3.322.1) of Gradshteyn & Ryzhik (1980) was found not to need its stated restriction $u > 0$, and an information-theoretic reason was found for time seeming to run backwards for Black & Scholes (1973).

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