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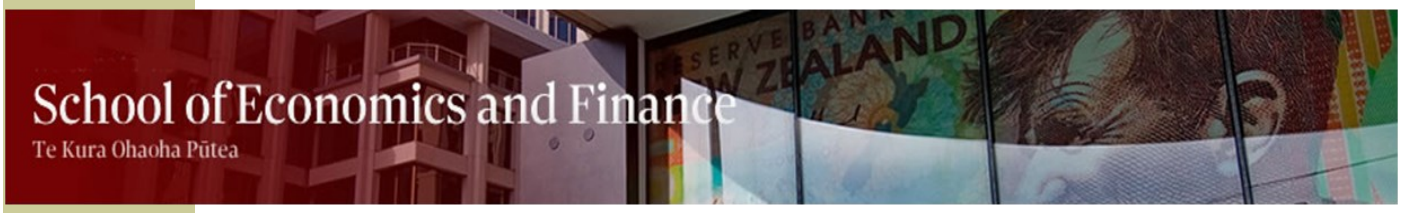

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SENSITIVITY OF CAUTIOUS-RELAXED INVESTMENT POLICIES TO TARGET VARIATION

JARRED FOSTER AND JACEK B. KRAWCZYK

Abstract. This study builds on recent findings that target-based utility measures, used in the dynamic portfolio optimisation, deliver investment policies that can generate left-skewed payoff distributions. These policies can lead to small probabilities of low payoffs. This is in contrast to the classical portfolio optimisation strategies that commonly deliver right-skewed payoff distributions, which imply a high probability of losses. The left-skewed payoff distributions can be obtained when a "cautious-relaxed" investment policy is applied in portfolio management. Such a policy will be adopted by investors who are both cautious in seeking a payoff meeting a certain target, but relaxed toward the possibility of exceeding it. We use computational methods to analyse the effects of varying the target on the payoff distribution and also examine how the fund manager's explicit preferences, when they differ from the investor's, can impact the distribution. We found that increasing the target causes the distribution to become less left skewed. Lowering the target slightly, keeps the left-skewed payoff distribution albeit the mode diminishes. Decreasing the target substantially so it is below the safe investment payoff, changes the skew. Investor's payoff will not suffer even if the actual fund manager allows for their own utility in the optimisation problem.

1. Introduction

In a recent paper, Krawczyk (2008) uses a target-seeking performance measure with different slopes on each side of the target, to replace the "classical" risk-averse utility function usually utilised for dynamic portfolio management in papers such as Merton (1971). The target-seeking problems solved in Krawczyk (2008) and in this paper[1] concern a pension fund investor deciding how to allocate funds between a secure and a risky asset. By optimising a target-seeking performance measure, new 'cautious-relaxed' policies are obtained that can generate left-skewed payoff distributions, as opposed to the right-skewed ones that result from maximisation of an expected concave utility function. The aim of this paper is to see how the dynamics of this problem changes when the size of the target is varied and also when the investment strategy is implemented by a "manager" whose goals might differ from those of the investor's ("client").

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[1]This paper draws from and extends Foster (2011).

In our view, an analysis of cautious-relaxed policies and of the resulting left-skewed payoff distributions constitutes a step toward a better understanding of certain investors' behaviour. We contend that some investors (eg, pension optimisers) are not uniformly risk-averse but, asymmetrically, adamant to avoid low payoff realisations, yet not zealously seeking to maximise them. Utility measures that are unsymmetrical and target seeking are also seen in prospect theory, in papers such as Tversky and Kahneman (1992), Berkelaar et al. (2004) and Jin and Zhou (2008), where a target seeking agent has a kinked function. However, in contrast to the prospect-theory utility measures that are convex before and concave after the kink, the utility measures considered here are locally concave ie, concave before and after the kink. Also, generically, under prospect theory, the investor is less concerned about big losses, than one with a cautious-relaxed utility measure (see Azzato et al. (2011)). This adversity to big losses means that our investor would be more accepting of small losses ie, slightly less than the target, than one following prospect theory.

Our underlying non-symmetrical (with respect to risk) cautious-relaxed utility measures, which can capture agent's target-seeking behaviour, are non-differentiable and only locally concave. This prevents us from seeking closed-form optimal solutions. Our solutions will then be numerical and, hence, parameter specific. The hypothetical base-case scenario, which we will study, involves an initial outlay invested in a pension fund to grow and be collected as a lump sum after a given optimisation horizon. The investor has the choice between two assets, one being risk-free and the other a "volatile" risky asset with an expected return higher than the risk-free asset's. All financial parameters like the secure asset interest rate, risky-asset drift, volatility, etc. are assumed known and the allocation policy is a maximiser of a given utility measure. Notwithstanding the solution parameter-specificity, our analysis can be extended to other cases through the use of specialised software (see Azzato and Krawczyk (2008b).)

The decision of how to allocate a portfolio in a dynamic setup is one that has been solved on countless occasions with the most prominent paper being Merton (1971) where Merton built a continuous-time version of a model proposed in Samuelson (1969). These were the first pioneering papers in terms of dynamic portfolio management. This is different to papers such as Markowitz (1952), which deal with a static version of the problem as opposed to a dynamic one. All these papers focused on utility functions that are HARA[2], as opposed to the above mentioned prospect-theory convex-concave utility measures or target-seeking, seen here[3]. The exception being He and Zhou (2011a) who deal, in a static setting, with the problem of a loss averse-agent and introduce a loss aversion measure called the large-loss aversion degree (LLAD).

In Pliska (1986) an optimal portfolio is chosen by modeling security prices as semi-martingales. In Boda et al. (2004) Markov decision processes are used to maximise the probability that wealth exceeds a certain target. In Cairns (2000) a contribution rate is incorporated into the problem, meaning rather than allocating a set amount paid at the

[2]Hyperbolic absolute risk aversion.
[3]Also see Krawczyk (2001), Krawczyk (2005), Azzato et al. (2011), Krawczyk (2008).

beginning of the investment period, the agent continuously contributes to the fund. This is an idea that in the future could be incorporated into the setup used in this paper. There have been countless other models and methods used at solving such dynamic problems. These include using VaR and CVaR as constraints as in Yiu (2004) and Bogentoft et al. (2001), but the strategies found in these models still generate right skewed distributions, which are deemed not preferable by pension fund optimisers.

The problem of how a portfolio is allocated when using a utility function like those seen in prospect theory was solved in Jin and Zhou (2008). These authors solved the problem by splitting it into subproblems and found that the optimal strategy is one in which the investor takes on aggressive gambling policies. This can also be seen in Azzato et al. (2011) where a prospect theory utility function generates a more aggressive investment strategy for any level of wealth than a comparable strategy of "our" cautious-relaxed investor.[4] An impact of the investment horizon on the investment strategies of a prospect-theoretic investor is studied in Dierkes et al. (2010); they show, using a *probability weighting function*, that the attractiveness of more secure investments increases as the horizon shortens. [5] However, up until this paper, the research has focussed solely on one target for the kinked utility functions' investors, with none being performed on the effect of increasing or decreasing it.

Another aspect of this paper (and others involving the cautious-relaxed investor), which differentiates it from other research in this area, is the approach we use to compare and rank utility functions. The traditional method to decide on a utility function is based on psychological experiments and perceived preferences of individuals. This is then used to derive a functional form for the utility function and a portfolio optimisation problem is solved based on this measure and the obtained strategy is *optimal*. However, there seems to be little attention given to the actual outcomes that eventuate from the investment strategy, deemed optimal for the adopted utility function. One exception to this is the work done on the distribution builder in Goldstein et al. (2008), in which experiments are performed where subjects are allowed to build their desired pension distribution subject to a budget constraint. The results lead to distributions that are right skewed, but this could be due to the setting of a reference point at the amount guaranteed by the risk-free asset. In He and Zhou (2011b) quantiles are proposed as an effective way to evaluate the success and failings of a portfolio. Indeed, we contend that a better way to compare utility functions is to look at the distributions of outcomes that arise form following each optimal strategy. This way, a better idea can be gained for the level of success that is derived from the investment strategy that is produced from the utility function. Only then can one come to a conclusion about the appropriateness of each measure and its level of realism when compared with the actual strategies used by investors.

[4] A comparison of the outcomes of both investment strategies: prospect-theoretic and cautious-relaxed can be found in Azzato et al. (2011) where, for the assumed parameters, the distributions obtained by the prospect theoretic investor appear similar to the Merton investor's. This idea is implied by the fact that they are both right skewed with a high probability of loss.

[5] Somehow similarly, the cautious-relaxed policies, which depend on the time remaining until the end of the horizon, advocate more secure investments once the target has been reached.

In our model, the current fund value is diminished by a management fee, as if a manager charged for their services. This may mean that if the fund was actually controlled by a "manager" and not by the "client", the "manager" may want to modify the cautious-relaxed policy, to suit their utility. In the later part of the paper, we propose a model modification that incorporates the manager's preferences. In that part of the paper, the portfolio will be allocated according to a strategy that maximises a weighted average of both the manager's and the client's utility measures. We note that this introduces a principal-agent dynamics to the problem. It is found that the effect this has on fund distributions is not overly significant and in some cases may even cause some investor's payoff improvements.

The paper proceeds as follows. Section 2 introduce the setup for the problem and the methods used to solve it. In Section 3 we present an optimal solution to the Merton-investor problem and show the resulting utility function distribution; a cautious-relaxed investor problem is formulated and solved in Section 4. In Section 5 the effect of varying of the investor's target is analysed and the fund distributions are compared. Section 6 deals with the manager's incentives and assesses the resulting distributions when the manager considers their own payoff. Section 7 finishes with some concluding remarks.

2. Setting up the Problem

2.1. **The model.** Consider an investor with savings of x_0 that will not be required for a period of T years (for example they may be retiring in T years). The investor wants to collect a payoff from x_0, after the T year period, from a pension fund. The investment strategy is chosen in order to maximise a certain objective function that could be decided on by the investor.

The investor's portfolio consists of two assets one being risky and the other risk-free, as is commonly seen in the literature. It is assumed that there are no transaction fees, meaning that rebalancing the portfolio has zero cost. However, later, we will allow for a management fee that the fund administrator may charge.

The price of the risky asset $p(t)$ follows a geometric Brownian motion as follows:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dw \tag{1}$$

where dw is a standard Brownian motion. The α term is a constant and represents the drift of the asset, higher levels of this represent higher expected returns for the asset. The σ term is also a constant and represents the volatility of the asset. The risk-free asset's price $q(t)$ follows the process:

$$dq(t) = rq(t)dt \tag{2}$$

where $r < \alpha$ and $r, \alpha, \sigma > 0$. At any point in time $u(t)$ is the proportion of wealth invested in the risky asset hence wealth $x(t)$ follows the process:

$$dx(t) = (1 - u(t))rx(t)dt + u(t)x(t)(\alpha dt + \sigma dw) - U(t)dt \tag{3}$$

where $U(t)$ is the agent's consumption rate.

The investor needs to choose a strategy such that it maximises discounted expected utility. This could mean that (4) is maximised:

$$J(x_0, u) = \mathbb{E}\left[\int_0^T e^{-\rho t} g(U(t))dt \,|\, x(0) = x_0\right] \tag{4}$$

where $g(U(t))$ is the investor's instantaneous utility function, $\rho > 0$ is the discount rate and $\mathbb{E}$ is a mathematical expectation. Now, (4) can be augmented without any loss of generality to become:

$$J(x_0, u) = \mathbb{E}\left[\int_0^T e^{-\rho t} g(U(t))dt + e^{-\rho T} s(x(T)) \,|\, x(0) = x_0\right] \tag{5}$$

where $s(x(T))$ is the final payoff function.

For this study, $U(t) = 0$ meaning that the agent will make no withdrawals from the fund until time T.[6] Consequently, the pension fund investor's objective function becomes:

$$J(x_0, u) = \mathbb{E}[e^{-\rho T} s(x(T)) \,|\, x(0) = x_0]\,. \tag{6}$$

Note that as $e^{-\rho T}$ is a constant it can be omitted without affecting the result.

Now, we will also assume that a fee $\epsilon x(t), \epsilon > 0$ is to be charged at a continuous rate as if a "manager" was charging for their services. The inclusion of this fee changes the dynamics of the problem slightly, in that (3) is now altered to become

$$dx(t) = (1 - u(t))(r - \epsilon)x(t)dt + u(t)x(t)((\alpha - \epsilon)dt + \sigma dw). \tag{7}$$

So, the *efficient* bond rate is $r - \epsilon$ and the risky asset drift $\alpha - \epsilon$.

Therefore a pension fund investor needs to maximise (6) subject to (7) and also $x(t) \geqslant 0$ and $0 \leqslant u(t) \leqslant 1$. [7]

2.2. **Parameters.** As stated in the introduction, our solutions will be numerical and, hence, parameter specific. The hypothetical base-case scenario, which we study, involves an initial outlay $x_0 = \$40\,000$ invested in a pension fund to grow and be collected as a lump sum after a given optimisation horizon of $T = 10$ years. Table 1 shows the rest of the parameters[8] that are to be utilised for this study.

r	α	σ	ϵ
0.05	0.085	0.2	.005

TABLE 1. Chosen parameters

[6] $U(t)$ can also be made positive without any effect on the Merton investor's strategy, considered in Section 3.

[7] This means we forbid the act of short selling.

[8] This paper is similar to Azzato et al. (2011) in its choice of parameters.

With these parameters the fund will accumulate to $40\,000e^{(0.05-.005)10} = 62\,732.49$ if all money is invested in the risk-free asset. From this point on, this figure will be referred to as the guaranteed amount. It should be expected that a pension fund investor would require the pension yield to regularly surpass this value.

2.3. **SOCSol.** Due to the non-symmetric and non smooth nature of the cautious-relaxed investor's utility function, an analytical solution to the investment problem is not available. We get around this by generating the optimal investment strategies using a program called SOCSol, developed in Matlab and introduced in Azzato and Krawczyk (2008a). The program discretises optimal control problems and solves them using Markov Chains, meaning the usual requirement of a differentiable objective function can be ignored. See Azzato and Krawczyk (2008a), Windsor and Krawczyk (1997) and Azzato and Krawczyk (2008b) for more details on SOCSol.

Different possibilities for the movement of the risky asset are generated within SOCSol using Monte Carlo simulation. Given each optimal investment strategy, this can be used to produce the distributions of wealth at time T. The histograms shown later are the distributions of fund value given by 5000 simulations.

On the other hand, the classical, concave utility-function optimisation is amenable to a closed form solution and will be presented below.

3. The Merton investor

3.1. **The classical utiltiy measure.** An expected-payoff utility function is the classical portfolio performance measure, see Merton (1971) and Samuelson (1969). In this section, we assume that a pension-fund investor uses this function. We will present the (known) optimal investment strategy and analyse the resulting payoff distribution.

The choice of $s(x(T))$ in (6) is now the issue of concern. The classical *Merton investor* will use the concave utility function as follows (compare Fleming and Rishel (1975)):

$$s(x(T)) = \frac{1}{\delta}\left[x(T)\right]^{\delta}, 0 < \delta < 1. \tag{8}$$

Analytically, the optimal solution to this problem can easily be found as a solution to the Hamilton-Jacobi-Bellman equation. First, we define a value function for the portfolio problem:

$$V(t,x) = \sup_u J(x(t),u). \tag{9}$$

This gives the following Hamilton-Jacobi-Bellman equation:

$$\max\left[\frac{1}{2}u(t)^2\sigma^2x^2V_{xx}(t,x) + (r + u(t)(\alpha - r) - \epsilon)x(t)V_x(t,x) + V_t(t,x)\right] = 0 \tag{10}$$

with boundary condition

$$V(T,x) = \frac{1}{\delta}\left[x(T)\right]^{\delta}. \tag{11}$$

Maximisation of (10) gives

$$u(t) = -\frac{\alpha - r}{\sigma^2}\frac{V_x(t,x)}{xV_{xx}(t,x)} \tag{12}$$

substituting into the Hamilton-Jacobi-Bellman equation (10) implies

$$0 = -\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2}\frac{V_x^2(t,x)}{V_{xx}(t,x)} + (r - \epsilon)x(t)V_x(t,x) + V_t(t,x). \tag{13}$$

The boundary condition set by (11) suggests the functional form:

$$V(t,x) = f(t)\frac{1}{\delta}x^{\delta},\ f(T) = 0\,; \tag{14}$$

plugging this into (13) we are left with an ordinary differential equation:

$$0 = \left[\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2(\delta - 1)} + (r - \epsilon)\right] f(t) + f'(t). \tag{15}$$

This ODE is easily solved with a solution $f(t) = 1$ implying that $V(t,x) = \frac{1}{\delta}x^{\delta}$, substituting into (12) gives

$$u(t) = \frac{(\alpha - r)}{\sigma^2(1 - \delta)}, \tag{16}$$

which is the known optimal solution to the Merton investor's problem (eg, see Fleming and Rishel (1975)).

So, we see that the management fee has no impact on the optimal strategy and, more importantly for our study, this strategy, for the *dynamic*-portfolio investor, is practically *static*. Ie, it does not change with time and wealth. The same proportion of wealth is always invested into the risky and risk-free asset. This seems to be quite unintuitive in that you would expect an investor to change their strategy in order to adjust to the performance of the portfolio. For the remainder of this paper the investor using (16) will be referred to as the Merton investor.

3.2. **The distributions.** The main argument of this paper (and also that of the other publications on cautious-relaxed policies) is that the Merton investor's strategy frequently delivers payoffs that are *low*, often beneath the initial outlay. If so, while a lotto-player will be happy to employ it, a pension fund investor will *not*.

Let us now then calculate the payoff $x(T)$ distribution for the Merton investor and examine its distribution to justify the above claim. We find[9] that after fitting the optimal control

[9]Refer to the methods put forward in Kloeden and Platen (1992) and Annunziato and Borzì (2010).

(17) to the state equation (7), wealth $x(t)$ is a Geometric Brownian Motion that follows

$$dx(t) = Mx(t)dt + \Sigma\, x(t)dw \tag{17}$$

where $M = \dfrac{(r-\alpha)^2}{(1-\delta)\sigma^2} + r - \epsilon$ and $\Sigma = \dfrac{\alpha - r}{\sigma\,(1-\delta)}$. This means that for an initial value x_0, wealth $x(t)$ $t \in (0,\, T]$ is a log-normally distributed random variable with expected value

$$\underline{\mathrm{M}}(t) \equiv \mathbb{E}(x(t)) = e^{Mt}x_0$$

and variance

$$\underline{\Sigma}^2(t) \equiv \mathrm{Var}(x(t)) = e^{2Mt}x_0^2\left(e^{\Sigma^2 t} - 1\right).$$

We also know that the wealth is the stochastic process

$$x(t) = x_0 \exp\left(\left(M - \frac{\Sigma^2}{2}\right)t + \Sigma\, w(t)\right), \quad t \in [0\; T]\,. \tag{18}$$

Finally, the probability density function of wealth $x(T)$ is log-normal:

$$f_{X_T}(x; \underline{\mathrm{M}}_T, \underline{\Sigma}_T) = \frac{1}{x\underline{\Sigma}_T\sqrt{2\pi T}} \exp\left(-\frac{(\ln x - \ln x_0 - T(\underline{\mathrm{M}}_T - \frac{1}{2}\underline{\Sigma}_T^2))^2}{2\underline{\Sigma}_T^2 T}\right), \quad x > 0 \tag{19}$$

where $\underline{\mathrm{M}}_T$, $\underline{\Sigma}_T$ are $\underline{\mathrm{M}}(t)$, $\underline{\Sigma}(t)$ evaluated at $t = T$.

We know that a log-normal distribution is generically non-symmetric and, in our view, this causes a problem of using policy (16) by an investor seeking to avoid large losses.

Plugging in the values given in Table 1 to the Merton solution given in (16) and using a δ of 0.05 gives an optimal $u(t)$ of 0.921. This means that 92.1% of wealth at any time is invested in the risky asset. Now that all the respective parameters have been set, we can evaluate values for the mean and variance using the formulae given above These parameters yield a mean of 86 595.5 and a standard deviation of 55 041.91. These values suggest frequent payoffs below the initial outlay of \$40 000, even before we have analysed their actual distribution.

This investment strategy provides the distribution of payoffs shown in Figure 1. The blue line represents the integral of the theoretical PDF derived in (19), where the limits correspond to each bin in the histogram.[10] We see that the payoff is right skewed with a relatively high probability of either being below the original investment or below the guaranteed amount.

On the other hand, Figure 1 shows that the right-hand tail of the distribution extends to \$150 000 and beyond and there is a good probability of having a high level of wealth, at the end of the 10 years. However, the probability of wealth being less than the guaranteed

[10]The background histogram shows the distribution of the Merton-investor's fund yields obtained through Monte Carlo simulation in SOCSol, which we produced using the same method as for the cautious-relaxed policy payoffs discussed later in the paper.The agreement between the theoretical distribution and that produced by SOCSol provides encouraging evidence of the accuracy of our method of generating the payoff distributions.

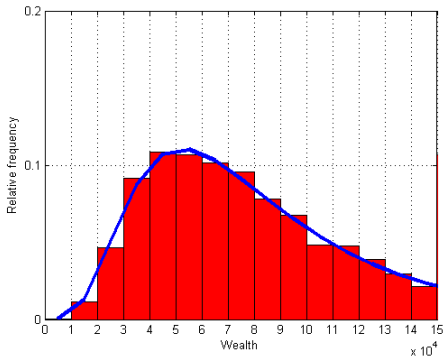

FIGURE 1. Payoff distribution for Merton's investor

amount is about 40%, hence making a loss appears worryingly frequent. Many agents saving for their pension would find this to be an unsatisfactory result.

4. The Cautious-Relaxed Investor

4.1. **A new measure.** This section will present an alternative, cautious-relaxed utility measure. Here, the investor sets a target level for the wealth at the end of the time period. If the wealth is below the target, the portfolio performance is quantified as profoundly negative (punishable) and any yield above it is moderately positive (rewarded). This is given using the following function:

$$s(x(T)) = \begin{cases} (x(T) - x_T)^{\kappa} & \text{if } x(T) \geqslant x_T, \\ -(x_T - x(T))^{a} & \text{otherwise} \end{cases} \quad 0 < \kappa < 1, a > 1. \tag{20}$$

Here x_T is a target set prior to the investment period, and the values of κ and a are set according to the level of reward or punishment for over- or underperformance, respectively. Note that the measure is concave both above and below the target (with a kink at the target). This is different to the prospect theoretic utility measures commented on in Section 1, in that they are convex below the target. A sensible target, x_T is selected so that the following inequalities are satisfied:

$$0 \leqslant x_0 < x_0 e^{(r-\epsilon)T} < x_T \tag{21}$$

where ϵ is the management fee. We will see in Section 5 that x_T may also be treated as a computational device, which can modulate the wealth distribution skewness. Currently, the inequalities are set as a matter of common sense as it seems unrealistic that an investor would set himself a target that is below the guaranteed level of wealth. The effect of relaxing these inequalities will be investigated later in this paper.

We notice that the target-seeking utility function in use here is punishing of any fund value below a set target and rewarding any amount above this target; however, the level of punishment for under-performance is greater than the utility gained by surpassing the target. This means that, with certain parameters, if the target is within reach all wealth

will be allocated to the risk-free asset. Otherwise, depending on the current fund position relative to the target, more or less is invested in the risky asset.

4.2. **Strategy and Distribution.** Now, we will call the investor that follows a policy that maximises (6) with (20) the cautious-relaxed investor (**CR**). The target, x_T, is originally set to \$100 000 because if the volatility from the risky asset is eliminated and only the drift remained then \$40 000 would become $e^{(0.085-0.005)10} = 89\,021.63$, which is "rounded up" to 100 000 to ensure it is not too easily obtainable. Finally, κ is set to 0.88 and a is is chosen to equal 1.5.

Investment strategies are generated by SOCSol on the discretised time and wealth grid. Figure 2 shows these strategies[11] for an investor maximising a cautious-relaxed utility measure with the parameters given above. The vertical axis is $u(t)$, the proportion to be invested into the risky asset and the horizontal axis is the level of wealth $x(t)$. For the sake of brevity we plot only two lines representing the investment strategies at time 0 and time 6.

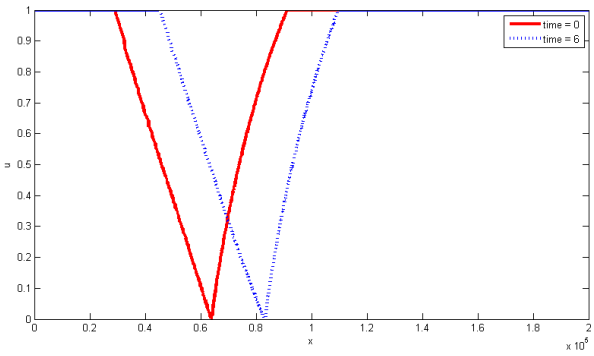

FIGURE 2. Control rule for target of 100 000

It is important to note that some levels of wealth, which we denote $x^S(t)$, will virtually generate no investment in the risky asset. This will happen when the target x_T can be obtained without the risky asset and if the risky-asset investment brings an expected value that is less than the secure investment. For example if at time 6, $x(6) = 83\,527$, then $u(\tau) \approx 0$ for $\tau \in [6, T]$. This is because $83\,527e^{(0.05-0.005)(10-6)} = 100\,000$ so, x_T is obtainable without the risky asset. For the relatively high volatility ($\sigma = 0.2$), the risky-asset investment of the size compatible with the adopted grid size, brings an expected value of marginal utility that is less than the secure gain and all funds[12] will be allocated to the risk-free asset. We can say that wealth at time 6 has reached the secure-investment level $x^S(6)$. Using $u(\tau) = 0$ for $\tau \geqslant 6$ from $x^S(6)$ causes every $x(\tau)$ to the right of the x-intercept of the strategy graph in Figure 2, to be reached at the "right" time for

[11] These are obtained in SOCSol using \$50 as the discretisation step of wealth and a time step of 0.01 of a year. The upper bound for wealth is set at 1 000 000.

[12] We can see in Krawczyk (2008) that $u(t)$ is never zero for less volatile risky assets.

the investor to continue the strategy $u(x^S(\tau)) = 0$. Investment strategies above each level $x^S(t)$ illustrate the investor's decisions if somehow, eg, because of a favourable noise realisation, the fund value were to be over this level.

The strategies that solve a portfolio-investment problem of the cautious-relaxed investor, shown in Figure 2 are dynamic: they change with time and wealth. This is different to the static strategy (16) obtained for the Merton investor.

More in detail, we notice that the investment minimum, which occurs at $u = 0$, defines two investment zones. (1) Fund values $x(t) \in (0, x^S(t))$, from which the target can only be reached by investing in the risky asset. Here, the investor must gamble to evade heavy penalties for falling short of the target. (2) Fund values $x(t) > x^S(t)$, from which the target can be reached by investing solely in the secure asset. In this zone, the investor maximises their reward for exceeding the target.

Despite this not being the norm in the literature, it seems intuitive that an investor would adjust their allocations with performance. For example, if an investor's fund is performing poorly, one would expect that the investor would be well advised to shift their investment towards more risky investments in order to at least have the chance of recouping losses.[13]

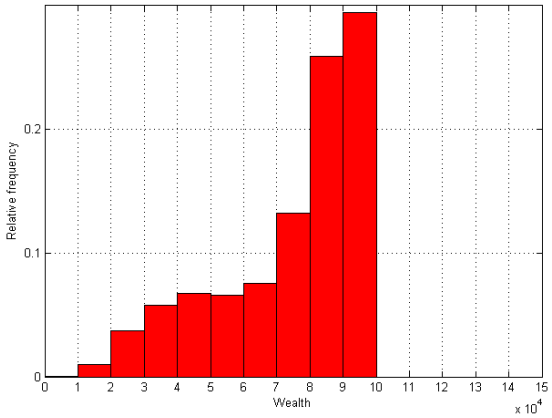

FIGURE 3. Payoff distribution for target of 100 000

Figure 3 shows the payoff distribution for an investor using a cautious-relaxed utility function with a target of \$100 000. We propose that this distribution should be considered favourable to the Merton investor's in that it is left skewed and the probability of loss seems to be much lower, which many investors would prefer.

This preference for left, or negative[14], skewness is related to the third derivative of the cautious-relaxed investor's utility function, which is negative when $a = 1.5$. A negative

[13]We abstract from the situations, in which the risky asset started performing poorly because the drift α decreased or σ changed.

[14]Many authors say that: • when the *left* tail is longer and the mass of the distribution is concentrated on the right of the figure, the distributions is said *negatively* skewed; • when the *right* tail is longer and

third derivative implies a dislike towards the third moment of a distribution ie, skewness or, in other words, preference for *negative* (or left). This goes against the norm in the literature, in which it is often suggested that an investor prefers positive skewness. Bali et al. (2011) and Barberis and Huang (2008) find that cumulative prospect theory can explain investor's desire for positive skewness, which can lead to right skewed securities being over-priced. This is because these investors are drawn to the lottery-like characteristic of these securities in that in odd cases they can provide extremely high returns. We propose that pension savers do not belong to this class of investors.

Chiu (2010) observes that gamblers appear to be skewness lovers, whereas we consider a pension-fund investor to be the opposite or an "antigambler". We believe that the high probability of loss, which can be associated with positively skewed distributions, would be unacceptable to such an investor even with the chance of big gains. Therefore we propose that a pension fund optimiser should have a preference towards negative skewness. This idea is supported by findings in Brockett and Kahane (1992), in which it is found that there are cases where a risk averse investor prefers negative skewness. This is not the case with both the Merton utility function and the function from prospect theory where the utility measures have positive third derivatives. This implies that the utility measure shown in (20) better captures the preferences of such a pension-fund investor towards negative skewness. We observe a clearly negative – ie, left – skewness in Figure 3, which as said above shows the pension-fund payoff distribution with target 100 000. We read from the histogram that the probability of a fund value below the guaranteed amount is approximately 25% for the cautious-relaxed strategy, 15% less than the Merton investor.

4.3. **Comparison of Distributions.** Table 2 (in Section 5.2) shows various distribution characteristics for these two investors under the "Merton" and "100,000" columns. (The other columns will be commented on in Section 5 when the savings target is varied.) The Merton distribution scores better than the cautious-relaxed, *only* with regard to the mean. The Merton distribution predicts higher probabilities of low payoffs (eg, 15% of receiving less than \$40 000 as opposed to 10% given by the cautious-relaxed policy), and a lower probability of scoring more that \$75 000. That feature may be crucial for a pension-fund investor to prefer the cautious-relaxed policy over Merton's, even though Merton's strategy has the potential to provide really high payoffs.

These results imply that the Merton investor's mean could be driven by very high outliers, which counteract a high probability of loss. In general, characteristics such as the median, the standard deviation and the probability of loss are also important to consider. These statistics will give a better idea of what is causing the mean to be at the level that it is. This is where the cautious-relaxed investor's policies in Krawczyk (2008) show some favourable (for pension optimisers) characteristics as the median is higher, the standard deviation lower and probability of loss is smaller.

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the mass of the distribution is concentrated on the left of the figure the distributions is said *positively* skewed.

The Value at Risk (VaR) and Conditional Value at Risk (CVaR) will also be taken into account as measures of success for each distribution. ${}_{\beta}$VaR for a given probability level β is defined to be the lowest level of loss such that the probability of making a larger loss is β. In our scenario, loss will be defined as any pension value below the initial outlay. ${}_{\beta}$CVaR is the the mean level of loss above the ${}_{\beta}$VaR. See Rockafellar and Uryasev (2000) or Bogentoft et al. (2001), also Krawczyk (2008), for more information on these measures. After looking at the characteristics for each distribution, a discussion of a pension fund investor's motivations can take place with the goal of analysing the effects of varying the target.

Notwithstanding the informativeness of the above *parametric* measures, it is the underlying payoff distribution, which caries full information relevant to the investor, which we propose to utilise to determine the utility function which is "right" for a pension saver.

5. Varying the target

5.1. **Strategies.** The question to be asked now is what is the best target for the cautious-relaxed investor to set, in order to get the most preferable payoff distribution? We suggest that they would be willing to accept a lower mean payoff for the sake of a lower variance and negative skewness, similar to that shown in Figure 3.

In Figure 4 the investment strategies for investors setting targets of \$80 000, \$120 000, \$140 000 and \$160 000 are shown respectively. These appear to be similar to those shown in Figure 2 but are shifted left for lower and right for higher targets. This is due to the differing levels of aggression required in order to reach more ambitious targets. This means that at any time and wealth level a higher value of $u(t)$ is required to reach the higher targets. Varying the target does not only shift the investor's strategy, the slope of the rules becomes flatter with higher targets and steepens with lower targets. This means that an increase in wealth has a more significant impact on the investment strategy for a lower target than one that is greater.

5.2. **Distributions.** Again, using Monte-Carlo simulation, the distribution of payoffs generated by the strategies shown in Figure 4 are produced, giving the histograms shown in Figure 5. The various parametric measures of "success" for the different utility savings' targets are shown in Table 2. The best value under each criteria is shown in bold font.

This table communicates some interesting results. Firstly, the mean always increases with the target but at a decreasing rate. The mean for a target of \$160 000 is \$83 310 which is relatively close to the mean of the Merton investor \$86 308. The median is highest for a target of \$120 000 and it begins to decrease after this; noticeably, all of the targets have higher medians than the Merton investor, which indicates left skewness. The 40th percentile[15] peaks with the \$100 000 investor and begins to drop as the target gets higher. The standard deviation of payoffs increases with the target implying that the higher the

[15]The 40th percentile is the value at which 40% of fund values fall below.

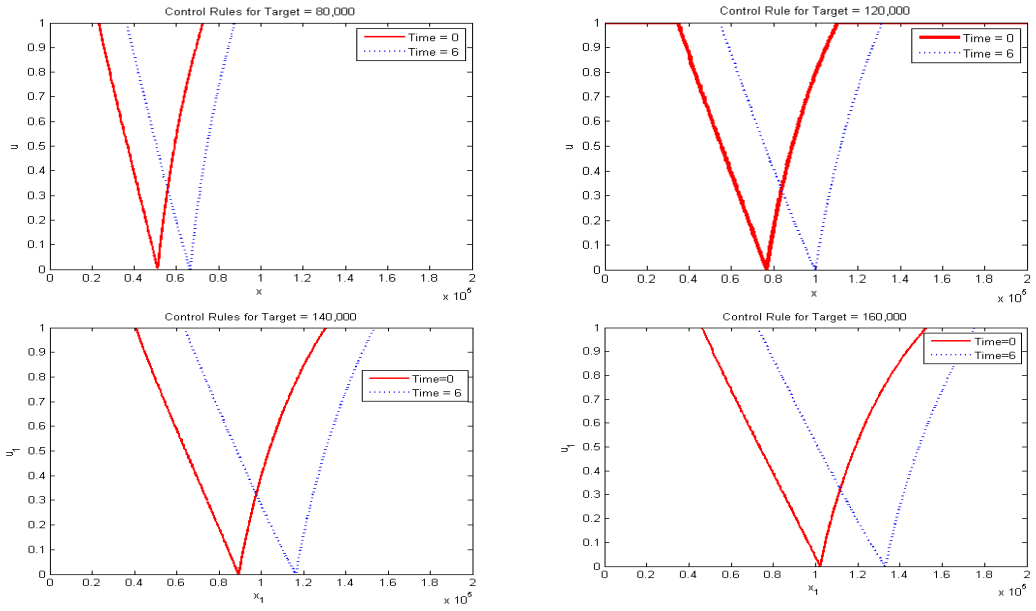

FIGURE 4. Portfolio management strategies for times 0 and 6.

Target	Merton	80 000	100 000	120 000	140 000	160 000
Mean of $x(10)$	**86308**	69 434	75 228	78 777	82 081	83 310
Median of $x(10)$	73 207	73 618	83 343	**87 306**	82 389	77 190
40-th percentile of $x(10)$	63 092	71 700	**78 119**	74 131	68 088	64 065
Std. dev. of $x(10)$	53 737	**11 231**	21 331	29 408	35 517	40 539
Coeff. of skew. of $x(10)$	1.8789	**-2.1697**	-1.0518	-0.4942	-0.1180	0.1696
5% VaR	11 304.78	**-25 25.47**	97 60.09	13 121.55	13 806.04	14 248.73
5% CVaR	17 074.97	**87 68.84**	16 146.53	18 819.93	19 205.84	19 851.18
10% VaR	5 101.34	**-15 305.95**	-328.77	6 067.88	6 062.49	8 302.43
10% CVaR	12 531.69	**-3 040.65**	10 496.73	10 742.56	14 462.86	15 538.68
$P(x(10) > .95x_T)$	N.A.	**0.3072**	0.1240	0.0572	0.0296	0.0186
$P(x(10) > \$75\,000)$	0.4848	0.3968	**0.6436**	0.595	0.5460	0.5182
$P(x(10) < \$62,732)$	0.3964	**0.1742**	0.2482	0.3242	0.3556	0.3876
$P(x(10) < \$40\,000)$	0.1494	**0.0414**	0.0984	0.1456	0.1530	0.1706
$P(x(10) < \$20\,000)$	0.0116	**0.0034**	0.0108	0.0164	0.0170	0.0194

TABLE 2. Final fund return distribution statistics. Highest ranked statistics are in bold font.

goal the more volatile the distribution. This is very much an expected result because the higher the target the higher the proportion allocated to the more volatile risky asset. However, the highest standard deviation of \$40 539 pales in comparison to the standard deviation of \$53,737 belonging to the Merton investor.

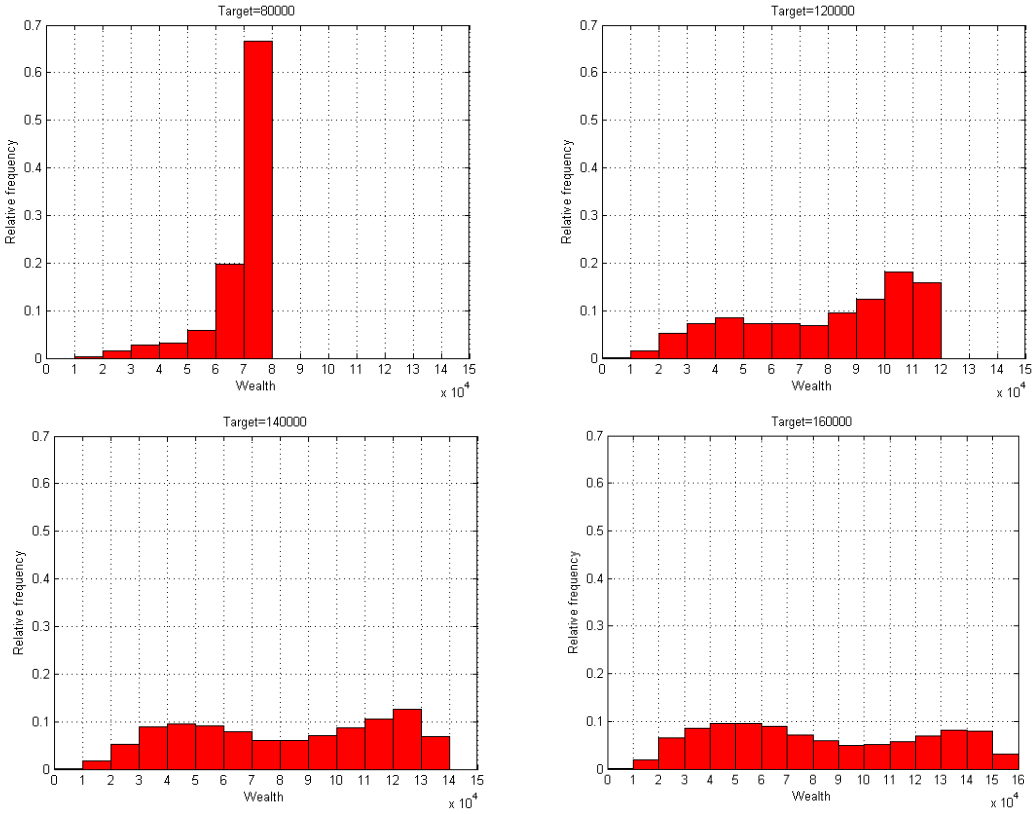

FIGURE 5. Portfolio payoff distributions at time 10

The skewness coefficient[16] is highly negative for lower targets, and increases with higher targets, eventually becoming positive for a target of \$160 000. This means the property of left skewness, claimed favourable, begins to diminish with higher targets and eventually the distribution becomes right skewed, similar to the Merton investor's. The Value at Risk (VaR) and Conditional Value at Risk (CVaR)[17] for both a 5% and 10% confidence level at first tend to increase significantly with the target but then appear to plateau for targets above \$120 000. It is interesting to see that both the VaR and CVaR for targets above \$100 000 are higher than that of the Merton investor, and that the 10% VaR for the \$140 000 target is lower that the \$120 000 target. In fact, it seems as if VaR and CVaR are indicators, for which the cautious-relaxed policies generate loss-prone distributions, especially with higher targets. This could be due to the fact that when these levels of loss are being made, the investor shifts all funds to the risky asset, which causes more money to be put at risk.

[16]The skewness coefficient is calculated as $E\left[\frac{(X-\mu)}{\sigma}^3\right]$ where μ is the mean and σ is standard deviation. It provides a measure of asymmetry in the distribution.

[17]${}_\beta$VaR for a given probability level β is defined to be the lowest level of loss such that the probability of making a larger loss is β. ${}_\beta$CVaR is the the mean level of loss above the ${}_\beta$VaR. In this scenario, loss is defined as any pension value below the initial outlay.

As the target becomes more difficult to obtain the probability of being within 5% of it decreases. An interesting characteristic to note is that the Merton investor has a higher probability of having a fund value below \$62 732 but he has a better probability of providing a fund value of above \$40 000 than both the \$140 000 and \$160 000 targeting investors. This could again be due to the fact that the target seeking investors, upon reaching the point where they are near \$40 000 are so far away from their target that they allocate all funds to the risky asset, which can be detrimental to their fund value.

As predicted, in many ways the investor targeting \$160 000 is very similar to the Merton investor in the characteristics of their distributions. The preferable properties of the investor with a \$100 000 target's distribution begin to get lost with these higher targets and the differences between them and the Merton investor begin to blur. However, it is important to consider the fact that the target-seeking investors statistics are affected a lot less by outliers. A risky asset, which is performing strong, can cause the value of the Merton fund to tend towards extremely high values causing big increases in the mean. This is not the case with the target seeking investors as once these funds are within their target, the investor will choose not to let them increase in value to the same extent.

5.3. **Which to Choose?** Looking at Table 2, \$80 000 has the highest number of criteria in bold font. This is mainly due to the fact that the target is relatively unambitious, and when it comes to probability of loss this target performs very well. However, in terms of mean and median this target performs the least favourably. This could potentially lead one to select \$100 000 as a target, where the 40th percentile and the probability of being greater than \$75,000 are in bold. This has come at the cost of slightly higher probabilities of loss, but with a higher mean and median than the \$80 000 target. Further increases in the targets cause gradual worsening in these statistics, therefore \$100 000 seems to be the most suitable target.

Figure 6 shows the PDFs and CDFs of the different investors. These are generated using the Parzen-Rosenblatt window method, which is a convenient way of smoothing the data and is often considered an improvement on the jaggedness of the histograms shown in Figures 5 and 3.

As the target increases the PDFs begin to flatten out. An interesting feature in the PDFs is the significant drop in the peak from the 80 000 target line to the 100 000 line, showing how much more likely achieving a target close to 80 000 is than the other targets. The CDFs act as would be expected, also becoming less and less steep as the target increases. It is clear, simply from looking at the CDFs that no distribution dominates the others in terms of first-order or second-order stochastic dominance.[18]

[18]It is shown in Krawczyk (2008) that there is no first or second-order stochastic dominance between the Merton investor's distribution and cautious-relaxed.

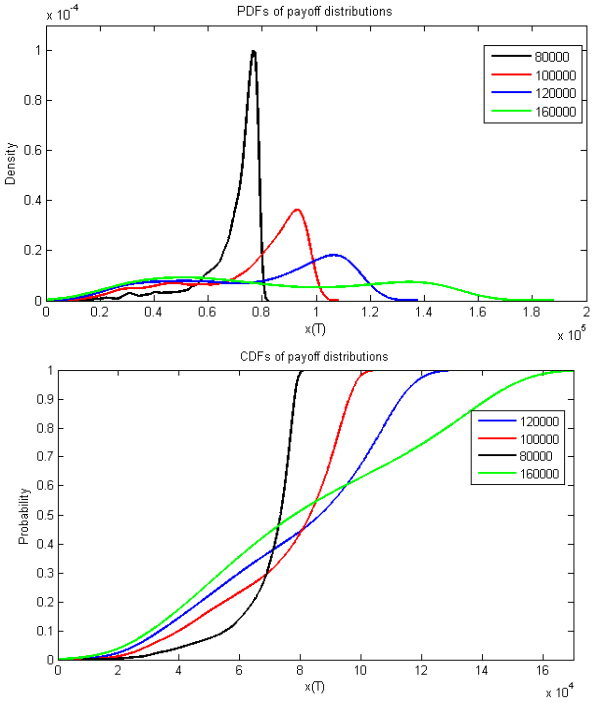

FIGURE 6. PDFs and CDFs for final distributions.

5.4. **A Lower Target.** We will now show that it is possible to devise portfolio management strategies, which generate positively (ie, right) skewed strategies with zero probability of loss.[19]

Consider loosening the inequalities set in (21) to check the effect of setting the target below the guaranteed amount $x^S(0)$. This would imply that the investor starts investing to the right of the x-intercept at the beginning of the investment period. This is because the target is guaranteed at this point, meaning that some of the wealth can be risked by investing in the risky asset. If the risky asset performs poorly, wealth is allocated back towards the risk-free asset. However, if the risky asset performs well a larger proportion of funds are allocated to it. This means that the client is effectively setting a minimum amount that they will allow the value of their investment to fall to. For this example the target will be set to \$60 000 and \$50 000 which is below the guaranteed amount of \$62 732. The investment strategies and fund distributions are shown in Figure 7.

This gives right skewed distributions, but unlike Merton's model the probability of being below the original outlay of \$40 000 is zero. The shift in attitude towards skewness is due to the fund predominantly being above the target meaning the third derivative of the cautious-relaxed utility measure is positive implying a preference towards skewness. Some pension fund investors could be drawn to this distribution, because of the fact there is a probability of some higher payoffs without the risk of significant loss shown in the other distributions. Note that this changes the dynamics of the investor's behaviour from

[19]Remember, the Merton investor's strategies are right skewed with a high probability of loss.

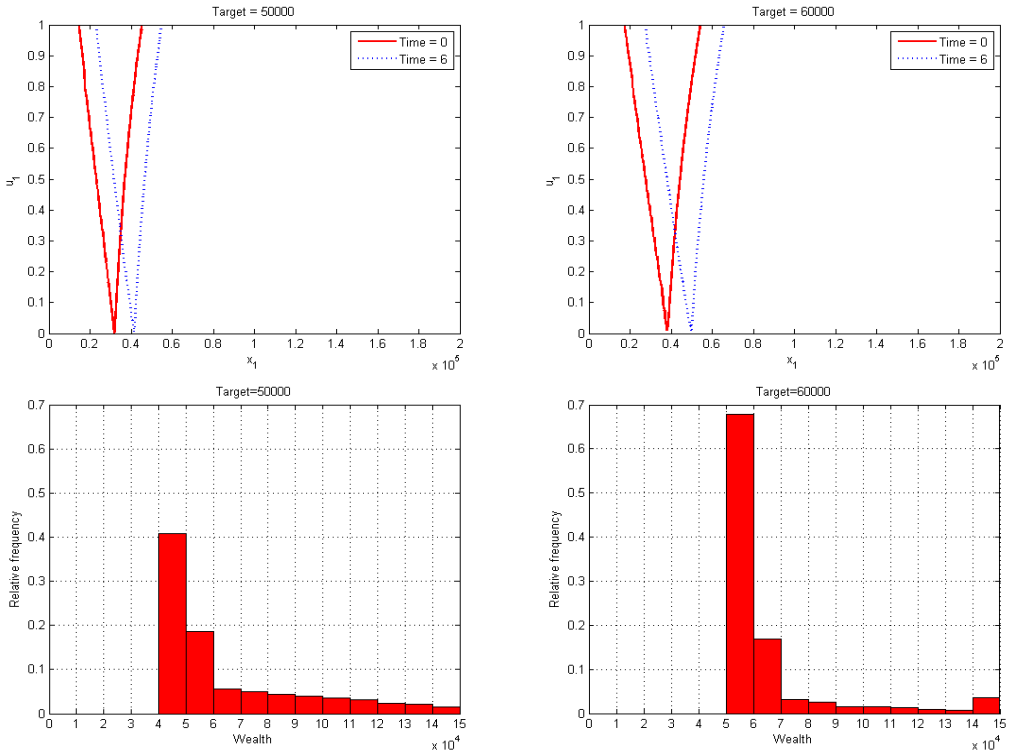

FIGURE 7. Histogram and control rules for targets of 50 000 and 60 000

one that is 'loss avoiding' to one in which they are setting a minimum value for the fund to fall to. The performance of this investor is still not competitive with the others in that the moment the portfolio begins to perform poorly all wealth is allocated to the risk-free asset and kept there for the remainder of the investment period. This is shown in the histograms by high frequencies of values at or close to the target. Like the Merton investor the performance of this fund is driven by outliers, which can provide misleading statistics.

6. The Manager's Incentives

In the previous sections it is assumed that the manager will be willing to follow the choices of the loss avoiding client, with no concern for the effect that this could have on their own payoff. It is possible that the manager will not be satisfied with the customer's decision and be tempted to deviate from it.

6.1. **Introducing the Manager's Utility.** In preceding sections the expected value of the final payoff function has been maximised without consideration of the manager's motivations. The effect of the manager deciding to shy away from the preferences of the customer towards their own payoffs is to be tested. This will be completed with the intention of testing the consequences that this may have on the investment strategy

and distribution of the fund. It will be assumed that the manager will now maximise a weighted average of their own and the customer's utility function. This would mean that the following equation would be maximised instead of (6):

$$J(x_0, u) = \mathbb{E}\left[(1-\nu)\int_0^T e^{-\rho t} f(x_m(t))dt + \nu e^{-\rho t} s(x(T))|x(0) = x_0\right] \tag{22}$$

where ν is the amount of weight the manager places on the customer's preferences and $f(x_m(t))$ is the manager's instantaneous utility function with regards to the manager's revenue $x_m(t)$. This implies that the closer ν is to 0 the more the manager is thinking about their own payoff in the present.

It will be assumed that the manager is risk neutral so that their instantaneous utility function is

$$f(x_m(t)) = x_m(t) = \epsilon x(t) = 0.005x(t). \tag{23}$$

This choice is made because the management fee is relatively low in comparison with the actual fund value. This means that modelling the manager as risk averse would cause their utility function to be even more dwarfed by the measure of the customer. The choice of $f(\cdot)$ implies that the manager's preferences are the opposite of the highly risk averse client's. The higher ν is, the lower the level of self-control the manager has and the more likely they are to invest in a way that differs to the interests of the client. Note that the discounting term has been added back into the customers utility measure in (22) because now the manager's utility is being discounted, so too must the customer's in order to be consistent. The problem to be solved now is to maximise (22) subject to (7) and other constraints mentioned in Section 2.1. The target is to be fixed at \$100 000.

The two extreme cases: of $\nu = 0$, when maximised is the expected value of the integral of the manager's utility function and of $\nu = 0$, when utility of the final clients's payoff is maximised, are different optimal control problems. However, one and the other concern $x(t)$, $t \in [0, 10]$ and, in broad terms, if one problem's performance index is high, so is the other's. For example, the manager will accomplish a great payoff only if process $x(t)$ realisations have been high; this forecasts a large final payoff to the client, if there was no substantial crash at the end of the horizon. On the other hand, if the clients payoff is large, then the $x(\tau)$ values for $\tau < T$ must have been high, unless there was a strong positive shock of wealth close to T. So, even if the two problems are dissimilar, a mixed criterion like (22) should not produce dramatically different investment strategies than the cautious-relaxed ones.

Furthermore, actions of the risk neutral manager may compensate the client's reluctance to go for big payoffs. Thus, it is not impossible that the resulting payoff distributions could be left-skewed with even less mass in the left tail, meaning less possibility of low payoffs. Below, we show how these predictions compare with "reality" ie, our solutions obtained from SOCSol by the Monte Carlo simulations.

Values utilised for ν start at 0.5 and are decreased from there. The first value for ν that has any visible effect on the investment strategy of the manager is 0.0001. This is so

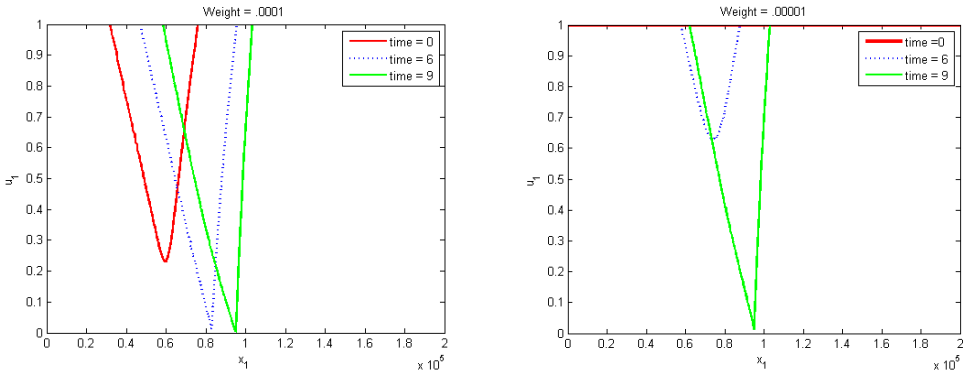

FIGURE 8. Control rules for managers with $\nu = 0.0001, 0.00001$

because the management fee is very small in comparison with the fund value.[20] We note the manager's strategy when ν is set to 0 is to allocate all money to the risky asset, i.e. set $u(t) = 1$, this is due to the assumption of risk neutrality. The investment strategies for $\nu = 0.0001, 0.00001$ are shown in Figure 8.

These control rules show that the manager's strategies tend away from those shown in Figure 2 towards these more aggressive ones, the lower the level of ν is set. The change to the strategy at time 0 occurs first, as at this point the manager's payoff is discounted the least, and therefore the reward is higher to the manager. Here, in Figure 8, a strategy for time 9 is included because it better illustrates the changes in strategy over time (which was unnecessary in Section 4 where the strategy slopes do not change much). The problem has changed in that the timing of wealth gains is now important to the manager and their tendency is to be more aggressive at times closer to 0 than at times further away. A good example of this is that the shape of the strategy at time 9 does not visibly change with the increase in ν. This is in contradiction to what is seen in earlier sections in that the client's preference is to be more aggressive in later time periods than earlier ones. Here, these two are effectively counteracting each other in that while the client wishes to become more aggressive later in the investment period, the manager's tendency is to take more risk in earlier stages of the investment period. Effectively, the utility measure used by this manager is one that is also target seeking, but where the penalty for failing to reach the target is not as harsh as the cautious-relaxed one used in this paper and, also, there is more reward for exceeding the target. It may hence be said that the utility function (22) reflexes preferences of an agent who is *less* cautious-relaxed than that introduced in Section 4.

6.2. **A Better Distribution?** The next question to ask is what effect do these new strategies have on the distributions of the fund? Figure 9 shows the histograms of the

[20]This level of weighting may seem small but it is important to remember that the manager only receives 0.5% of the actual fund value. Thus the weight on their own payoff will have to be extremely high for it to be comparable to the client's.

customer's final payoffs for the two strategies alongside the distribution obtained previously in Figure 3. The summary statistics for these managers as well as a manager with $\nu = 1$ are shown in Table 3.

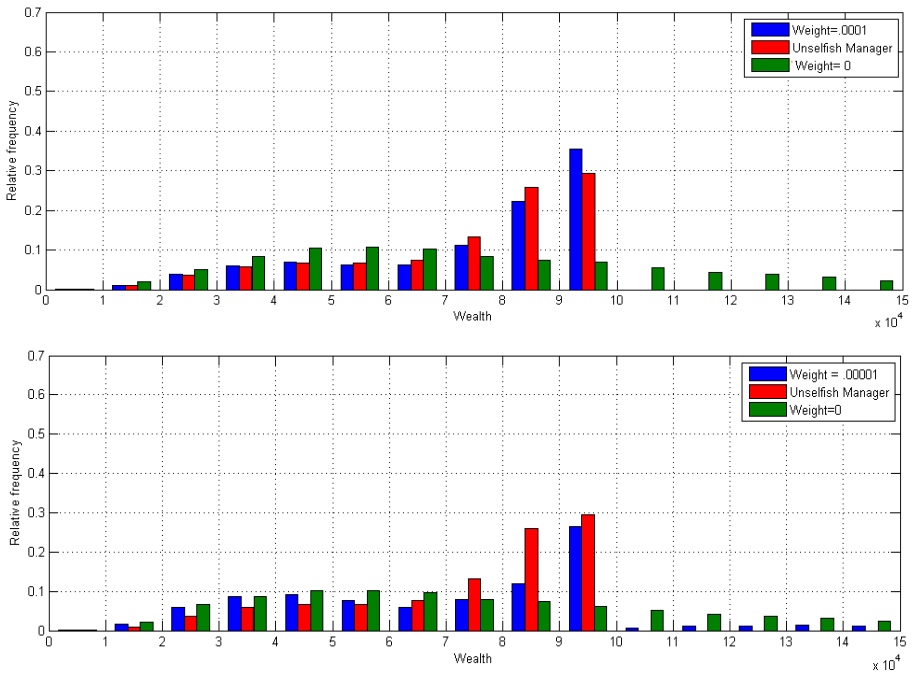

FIGURE 9. Wealth distributions for managers with $\nu = 0.0001, 0.00001$, found in the top and bottom panel, respectively

The distribution when ν is 0.0001 (hereafter the *selfish* manager's) appears to be superior to the distribution with $\nu = 1$ (hereafter called the unselfish manager's ie, "pure" cautious-relaxed). Evidence of this is that the probability of payoff being within the 90 000-100 000 bin is higher without significantly affecting the probabilities in the lower bins. When ν is equal to 0.00001 the histogram has reached values above the target; otherwise it has the rather familiar high mass in the left tail but with some local maximum about the initial value x_0. The case of a 'super'-sefish manager ie, when $\nu = 0$ generates rather flat distributions, with a right skew.

Table 3 shows[21] two of the better average fund values obtained so far. The mean of the manager with a $\nu = 0.00001$ (very selfish) is \$86,252, which is very similar to that of the Merton manager but with a higher median and a 5% smaller chance of making a loss. However, when compared with the unselfish \$100 000 targeting manager the standard deviation is significantly higher as too is the probability of loss. The manager with $\nu = 0.0001$ has a mean better than the \$100 000 targeting manager but this is counteracted with a slightly more volatile distribution and a higher chance of loss. The probability of

[21]Here we observe non-monotonicity of the skewness coefficient in ν. Despite this, the histogram for $\nu = 0.00001$ still reflects the favourable characteristics of previous negatively (left) skewed distributions.

ν	1 (**CR**)	0.0001	0.00001	0
Mean of $x(10)$	75228	75449	86252	**88532**
Median of $x(10)$	83343	**84860**	83183	74044
40-th percentile of $x(10)$	78119	**78458**	72003	62475
Std. dev. of $x(10)$	**21331**	22322	56190	59330
Coeff. of skew. of $x(10)$	**-1.0518**	-1.0000	2.7986	2.1023
5% VaR	**9760.09**	10094.47	14146.19	13853.81
5% CVaR	**16146.53**	16794.07	19484.15	19583.19
10% VaR	**-328.77**	1559.01	6623.27	7239.17
10% CVaR	**10496.73**	11213.83	14918.14	15099.21
$P(x(10) > .95x_T)$	0.1240	0.1722	0.3174	**0.345**
$P(x(10) > 75000)$	**0.6436**	0.6426	0.5782	0.4904
$P(x(10) < \$62,732)$	**0.2482**	0.2594	0.3450	0.4024
$P(x(10) < \$40\,000)$	**0.0984**	0.1112	0.1618	0.1690
$P(x(10) < \$20\,000)$	**0.0108**	0.0118	0.0178	0.0190

TABLE 3. Summary statistics for fund values where manager places weighting of $1-\nu$ on their own payoff. Highest ranked statistics are in bold font.

being within 5% of the target of \$100 000 is 5% higher than previously ie, when the investor was cautiously-relaxed (or, if "this" manager was unselfish and used $\nu = 1$). Note that the probability of making over \$75,000 has decreased and the probability of making under the secure amount has increased. This shows, as expected, that a risk neutral manager who starts to think of their own payoffs over that of their client will cause the mean of distribution of the fund to to increase at the cost of a higher level of volatility. These distributions still show some strong characteristics, however, and most likely the customer will not be able to spot the manager's deviation unless they are able to see the manager's investment strategy. Overall, the payoff distribution for $\nu = 0.0001$ displays the basic features of a left-skewed payoff distribution of a cautious-relaxed client, moderated by the risk neutral manager's policy.

6.3. **Sensitivity Analysis.** According to some performance measures and Figure 9 it seems as if the manager with a $\nu = 0.0001$ has actually improved the client's final payoff distribution by considering their own payoff. The volatility has increased slightly but the mean and probability of achieving the target have also increased. However, when considering the probability of making over \$75,000 the selfish manager does not perform as well, but only to a very small extent. This could possibly bring into question the quality of the utility measure $s(x(T))$ and how to improve it. To investigate this matter further, in this section a sensitivity analysis will take place. The goal being to vary the volatility parameter σ and testing the effect that this has on the performance of the two managers.

In order to make a fairer comparison of the two managers, the volatility of the risky asset will be varied. This will show how each manager fares with both a more and a less volatile

risky asset. One would expect to see the selfish manager provide a higher probability of loss with a more volatile asset as the strategy is more aggressive. On the other hand with a less volatile asset this manager could potentially provide a better payoff distribution. Figure 10 shows the investment strategies and final payoff distributions when the volatility of the risky asset σ is decreased to 0.05. The strategy of the selfish manager is omitted because $u(t) = 1$ for all values of wealth and time. Also for the unselfish manager, more is invested in the risky asset than when $\sigma = 0.2$; overall the problem becomes a close approximation to one that is deterministic rather than stochastic.

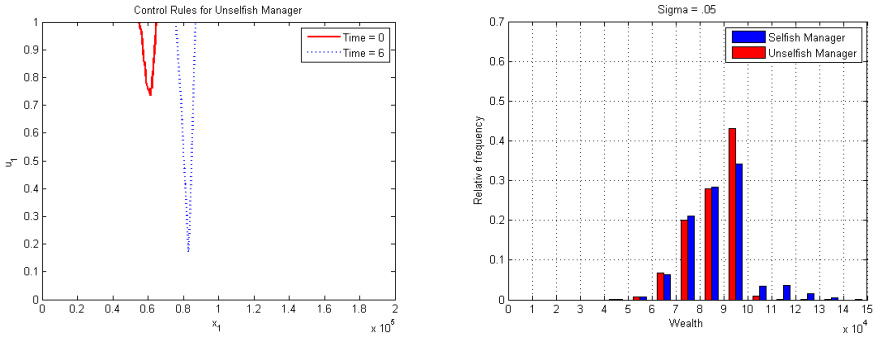

FIGURE 10. Control rule for unselfish manager and final distributions when $\sigma = 0.05$

Looking at the histograms one could potentially come to the conclusion that the selfish manager does perform better than the unselfish manager in this case. Above \$90 000, the selfish manager easily outperforms the unselfish manager. The probability that the client will receive a fund in excess of the target is much higher with a selfish manager. The median and VaRs of the selfish manager are also an improvement on the unselfish manager. Analysis, later on in the paper, shows the only drawback to be that the probability of obtaining over \$75 000 is slightly lower. These deficiencies are quite minimal but suggest that the cautious-relaxed utility measure may not be ideal when dealing with problems with low volatility σ hence almost deterministic. Graphs shown in Figure 14 demonstrate changes in the customary distribution-parameters obtained for the two managers for different volatilities.

We have aslo calculated the fund value at discrete time steps during the investment period. From this an approximate value of each manager's payoff throughout the period can be calculated by summing the revenues at each time step. This will give an idea of the improvement in the manager's payoff from the new strategy. The average selfish manager's payoff for a sigma of 0.05 is \$2 681 compared to \$2 670 for an unselfish manager. This shows that there has been an increase in revenue from the management fee, but a minimal one.

A possible method for identifying the differences between two investment strategies is plotting the paths or time profiles of both managers' portfolios. The graphs show how ten selfish and unselfish funds evolve over time, given the investment strategies shown in

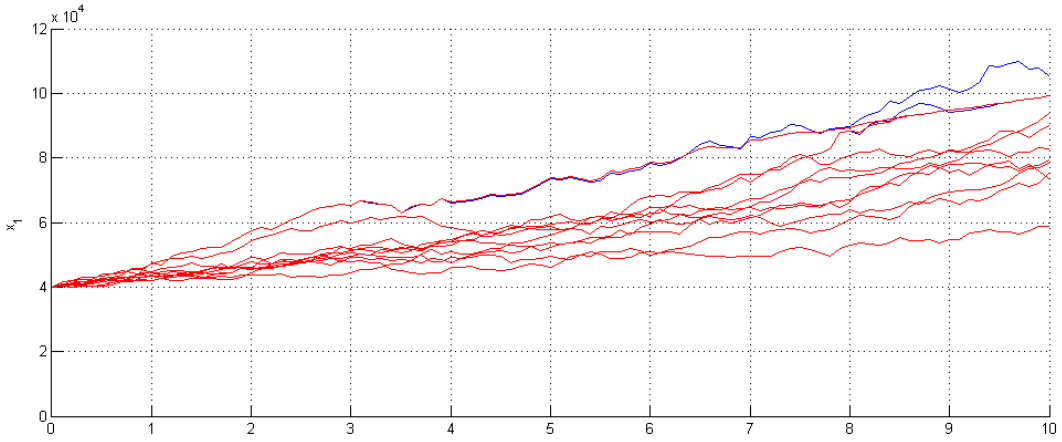

FIGURE 11. Wealth tme profiles for selfish (blue) and unselfish (red) manager when $\sigma = 0.05$

Figure 10. The results for both managers are simulated using the same random numbers, meaning the behaviour of the risky asset is identical for both of their portfolios. The time profiles of wealth when σ is 0.05 are shown in Figure 11. To help illustrate the difference between the two managers the simulations for both selfish and unselfish are shown on the same plot. The selfish manager's are shown in blue and the unselfish manager's are shown in red. Figure 11 shows the similarities between the wealth realisation profiles of the two managers at this level of volatility, in that all but two of the runs show the same fund evolution over time. Only in the case when the risky asset performs very well does the selfish manager start to excel. An interesting point to note is there seems to be little smoothing of the fund value by the selfish manager as one might expect. The manager's payoff is dependent on the fund value throughout the investment period, therefore it would be expected that there would be some incentive to attempt to raise the fund value during the investment period. But, for $\nu = 0.0001$ and $\sigma = 0.05$, the cautious-relaxed features of the investment policy appear to prevail and, as said, we observe little or no fund raises for most of the wealth profiles.

Now, moving on to a more stochastic scenario, the strategies and distributions of the two managers when $\sigma = 0.4$ are shown in Figure 12.

The selfish manager's strategy is slightly more aggressive and as a result shows a better probability of achieving the target. Whether the selfish manager's distribution is superior to that of the unselfish manager is again a matter of preferences. However, the probability of achieving a fund that is within 5% of the target has improved slightly. The probabilities of loss are only slightly higher (less than 1%) for the selfish manager, for approximately a 2% increase in the probability of achieving over \$75 000. Again, this could lead one to conclude that an improved distribution might be the result of a manager acting selfishly when the client has a cautious-relaxed utility measure. It is very difficult to make such claims because it is highly dependent on the choices of the client, however the manager

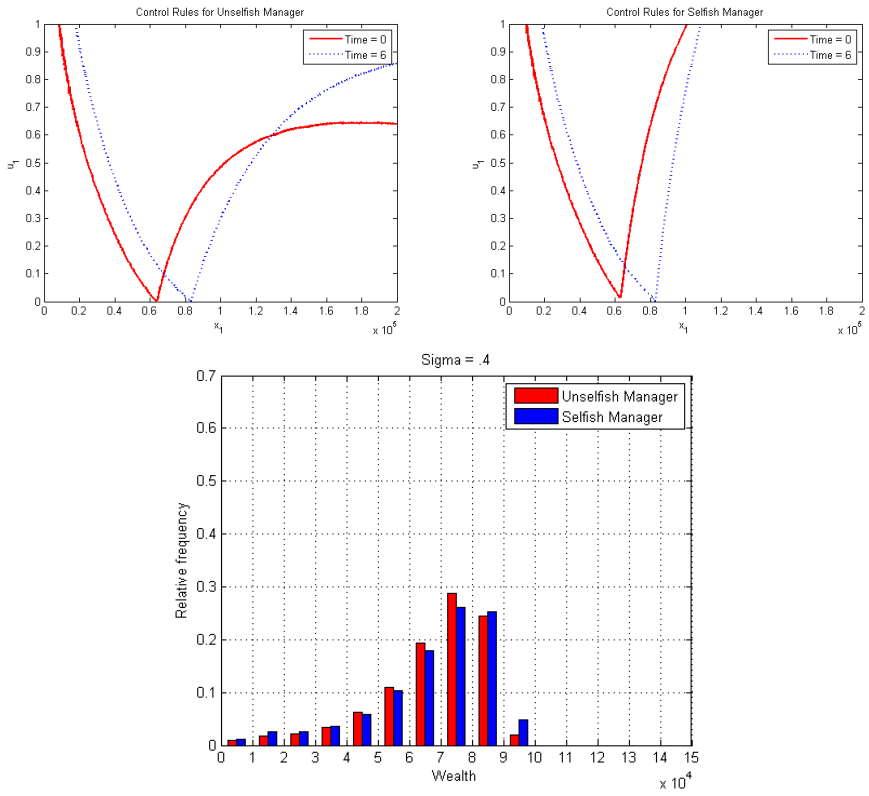

FIGURE 12. Control rule and final distributions for selfish and unselfish manager when $\sigma = 0.4$

acting in their own interest has not caused a decline in the quality of the distribution. The selfish manager's payoff is now \$3 104 compared to \$3 091 for the unselfish manager. This again shows a very small reward for the manager considering their own preferences, while allocating the *client's* portfolio. The corresponding time profiles are shown in Figure 13 and comparative statistics are plotted in Figure 14.

In this scenario the difference between the two strategies is a lot easier to see. The increased volatility is much more obvious here in that when the risky asset is performing strongly in the upper section of the plots, the selfish manager starts to perform better. In the lower section of the plot it is the unselfish manager who is more cautious, with the red lines lying above the blue. This gives a good illustration of the increased volatility brought out by the fund manager thinking of their own payoffs.

The plots shown in Figure 14 provide a summary of the above findings, with other levels of σ added to provide extra insight. The graphs show that there are not strong relationships between the measures and the two manager's performances. It is rare that one manager clearly outperforms the other in a measure for all levels of volatility. The selfish manager's mean and median is superior for all tested values of sigma 0.6 and below, above this level the unselfish manager's are better. The client's utility measure (the mean value of $s(x(T))$)

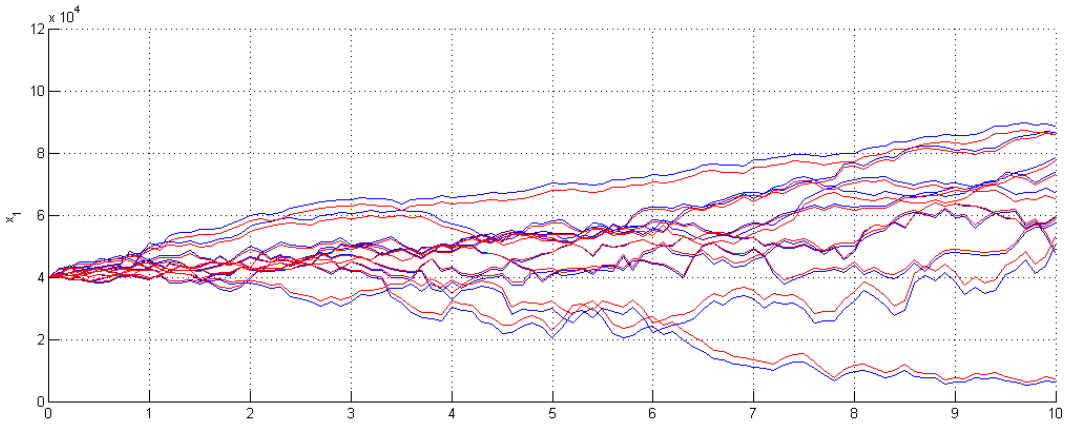

FIGURE 13. Wealth time profiles for selfish (blue) and unselfish (red) manager when $\sigma = 0.4$

of the unselfish manager is always higher as expected. The probabilities of loss (ie, the probability of making less than \$40 000 or \$62 732) seem to be consistently higher for the selfish manager. An interesting characteristic of these plots is that the probability of having a final fund value less than \$40 000 starts to decrease for higher values of σ. This is at first an unexpected result as you would expect with a higher volatility would come a higher probability of loss. However, looking at the investment strategies of both managers with high values of σ, we note that they are very cautious. For example the manager's both have approximately 20% invested in the risky asset at time 0 when σ is 0.4. This compared to around 60% for a sigma of 0.2. Finally, the probability of obtaining a fund value over \$75,000 appears better for an unselfish manager with a less volatile asset, but this is reversed when σ is at a higher level.

This analysis shows that a manager acting selfishly on their own behalf does not have large ramifications in terms of the performance of the fund. In some cases, certainly because of the impact of the neutral-risk part of the combined measure on the strategy, the performance of the fund seems to improve. This is to be especially true when the risky asset is less volatile in that there is an increased mean, without any significant decrease in the other characteristics. In this case perhaps the unselfish manager would be well advised to choose a higher target.

7. CONCLUSION

This paper performs a further analysis of issues brought up in Krawczyk (2008) involving cautious-relaxed investment strategies obtained as optimal solutions to a stochastic optimal control problem. It was shown there that an investor with the utility measure kinked around a target, also used in this paper, can find favourable left-skewed payoff distributions, which result from the kinked function optimisation. We claim and document in this paper that these distributions, and hence the underlying strategies, should be attractive

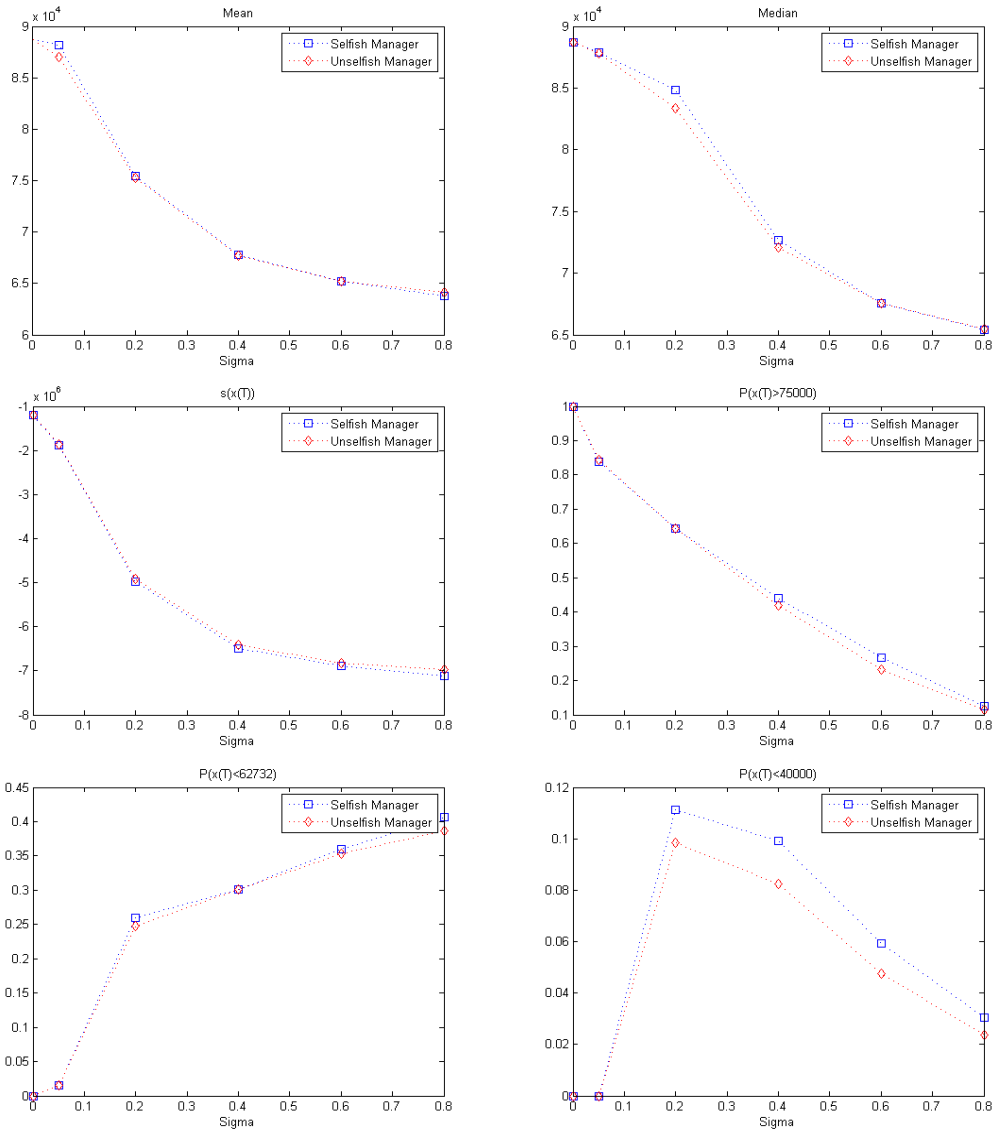

FIGURE 14. Comparisons of managers' statistics with various volatilities

to loss-avoiding portfolio optimisers like pension-fund savers. On the other hand, these savers might find undesired, the classical Merton-model based strategies, which generate right-skewed payoff distributions.

We have analysed the success of various investment strategies by simulating distributions of fund values using Monte Carlo methods. This means that a bigger picture is provided than when one simply looks at one or two performance indicators, such as the mean and/or variance. The use of histograms and a range of statistical properties allows us to gather a much wider view of the pros and cons of various investment strategies. While this can provide some great insight and we strongly believe it is a step forward in the field, a disadvantage is that it makes the strict ranking of utility measures very difficult. This

is because there are no guidelines on the exact properties that the investor does and does not desire.Rather the problem becomes one where numerous distribution characteristics are weighed up against each other resulting in a subjective opinion being made. The result of this being that our definition of the investor in this problem as one that is a pension-fund maximiser, heavily influences the ranking of portfolios in this paper.

Increasing the target causes investors to be more aggressive at the beginning of the saving period ie,, they commit more wealth to the risky asset when the target is large than when it is small. This riskier investment strategy leads to improved means and medians at the expense of higher variances and skewness coefficients. An argument can still be made that these distributions are an improvement over that of a Merton investor. The effect of setting a target below the secure amount is also assessed. Here, the resulting distributions possess some favourable qualities in that no fund value below the initial outlay will be obtained. However, this advantage is outweighed by too high a frequency of fund values below the guaranteed amount.

Further, we take a slightly different tact by attempting to model the incentives of the manager, and how this might affect the client's payoffs. It is shown that a risk-neutral manager who puts some weighting on their own payoff introduces slight improvements in a few parametric measures of the distribution. We show that the manager improves the mean and medians of fund distributions for most levels of volatility of the risky asset, without significantly affecting the probability of loss.

In summary, this paper has shown that findings in Krawczyk (2008) are robust in that variations in the size of the savings target provide distributions that can still be claimed better for pension savers than those found by the Merton manager. We have also provided evidence that this finding remains true, even when a manager is tempted to deviate towards their own payoff. Further study could assess the robustness of these findings to transaction costs as well as the incorporation of a jump diffusion element in the risky asset.

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