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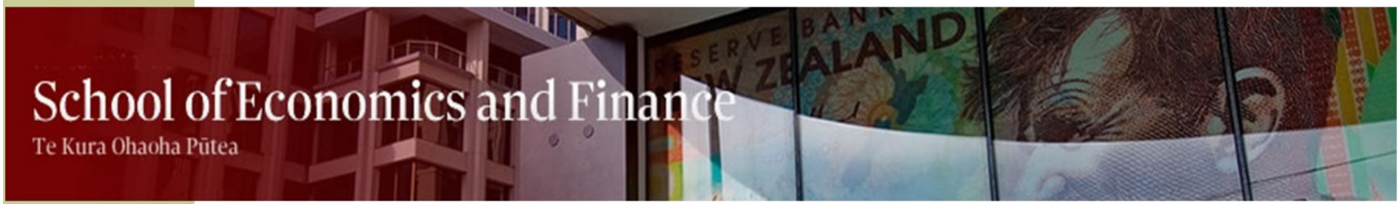

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Miller's (2009) WACC Model: An Extension

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Miller's (2009) WACC Model: An Extension

Abstract

Miller (2009a) presents an analysis of the weighted average cost of capital WACC model. The paper attracts debate which uses a variety of repayment schedules to support the arguments raised. We present an extension of Miller's (2009a) WACC model in a world where interest is tax deductible and debt principal is paid at maturity. We also present the corresponding model for the required rate of return on levered equity which is a vital input to the WACC model. Since these models are unwieldy, we explore an alternative definition of the WACC. These models provide insights into the debate on Miller's (2009a) paper.

Keywords: WACC, finite life, discount rate, tax shield, APV

JEL: G31, G32

Miller's (2009) WACC Model: An Extension

1. Introduction

Miller (2009a, p. 128) advances the thesis that the textbook weighted average cost of capital "*is not quite right*". He examines a number of repayment schedules to illustrate his argument. Bade (2009) and Pierru (2009a) take issue with Miller (2009a) and offer alternative insights and repayment schedules. Finally, Miller (2009b) offers a reply to Pierru (2009a) which is further debated by Pierru (2009b). This issue of repayment schedules is important to our understanding of the weighted average cost of capital model. However, a valuable insight by Miller (2009a) has not received the attention it deserves. Miller (2009a, equation (23), p. 135) derives what can be called the 'finite life weighted average cost of capital' model for the case where there is not tax relief on interest paid. His model differs from the corresponding textbook model.

Our contribution to the literature is to extend this work of Miller (2009a) by deriving the finite life weighted average cost of capital WACC model in a world where there is tax relief on interest paid. As with all studies of the WACC procedure, we start our analysis in Section 2 with a Modigliani & Miller (1963), hereafter MM (1963), world and derive the finite life version of the MM (1963) valuation model. This model is used in Section 3 to derive the finite life WACC model in the case where debt principal is paid at the maturity of the project. To permit the application of this WACC model, we derive the equation for the corresponding required rate of return on levered equity. Put succinctly, these two models are unwieldy. Thus, in Section 4 we take a subtly different approach to the modeling of the WACC. This approach requires the derivation of a finite life model of translating a levered beta to an unlevered beta. Section 5 provides brief concluding remarks and raises the issue of the relative merits of the WACC model compared to the APV model.

2. Finite Life Modigliani & Miller (1963) Model

The textbook Modigliani & Miller (1963) model is written as

$$V^L = V^U + B \times T_C \quad , \tag{1}$$

where: V^L is the market value of the levered firm, V^U is the market value of the unlevered firm, B is the market value of debt and T_C is the corporate tax rate. There are two definitional statements: (i), $V^L \equiv S^L + B$, where S^L is the market value of levered equity and (ii) $V^U \equiv S^U$ where, S^U is the market value of unlevered equity. The primary assumption is that these two firms operate in a world of taxation where interest paid is tax deductible.

The derivation of equation (1) is shown in most finance texts (e.g., Ross, Westerfield & Jaffe, 2010). The term $B \times T_C$ is the present value of the interest tax shield $INT_t \times T_c$ in a perpetual world, that is, $(INT_t \times T_c)/r_b = (B \times r_b \times T_c)/r_b = B \times T_C$. When a finite life is assumed, the present value of the tax relief is specified as $PV\left[INT_t\right] \times T_C$. The assumptions implicit in the $PV\left[\bullet\right]$ operator are unbiased expectations relating to the future cash flows and the appropriate risk adjusted discount rate. Thus, the finite life MM (1963) valuation model is written as

$$S^L + B \equiv V^L = V^U + PV[INT_t] \times T_C \quad . \tag{2}$$

It is a simple process to convert the finite life MM (1963) model to the equivalent Adjusted Present Value APV model (Myers, 1974). Consider a newly established unlevered firm owning a single asset. The value of this firm can be written as $V^U = Cost_0 + NPV_0^U$, where $Cost_0$ is the initial cash investment in the asset. The value of the comparable levered firm with the same asset is written as $V^L = Cost_0 + NPV_0^L$. It follows from substitution into equation (2) that

$$NPV_0^L = NPV_0^U + PV[INT_t] \times T_C \ , \tag{3}$$

which is the statement of the APV model. The model is devoid of additional assumptions.

The WACC method is a special case of the APV method, or the MM (1963) model (Miles & Ezzell, 1980, p. 720 or 727). A workable definition of the WACC is "The discount rate that converts the unlevered cash flows (see Miller, 2009a, p.130, footnote 4) of the project to the net present value calculated by the APV method". To illustrate this definition, consider a project with an initial cost of $Cost_0$ which generates unlevered and uneven post-tax cash flows X_t^U for *n* years. The general statement of the net present value model is

$$NPV_0^{Model} = \sum_{t=1}^{n} \frac{X_t^U}{(1 + r_{Model})^t} - Cost_0 \quad , \tag{4}$$

where the superscript (or subscript) *Model* represents either the WACC model or the unlevered model. The current market value V_0^{Model} is

$$V_0^{Model} = Cost_0 + NPV_0^{Model} = \sum_{t=1}^{n} \frac{X_t^U}{(1 + r_{Model})^t} \quad . \tag{5}$$

When $r_{Model} = r_e^U$, the required rate of return on unlevered equity, we get $V_0^U = Cost_0 + NPV_0^U$ and when $r_{Model} = r_{WACC}$ we get $V_0^L = Cost_0 + NPV_0^{WACC}$. Substitution into the finite life MM (1963) valuation model (equation 2) gives

$$NPV_0^{WACC} = NPV_0^U + PV[INT] \times T_C \quad . \tag{6}$$

Thus, the WACC method, the APV method and the MM (1963) model generate identical net present values in a finite world. The only assumptions are unbiased expectations relating to cash flows and discount rates.

The current orthodox opinion on the WACC is based on Miles & Ezzell (1980, 1985). They show that the textbook WACC applies in a perfect capital market under the assumption that the firm maintains a constant leverage over the life of the project. They state that the textbook WACC does not give the correct result if this assumption is denied. They show the linkage between investment and finance using a backward iteration procedure. An extension is offered by Harris & Pringle (1985) who derive a continuous-time version. Recently, Pierru & Babusiaux (2010) extend the application of the WACC model to the case where the firm capitalizes interest costs.

3. Finite Life Weighted Average Cost of Capital

The explicit additional assumptions are: (i) the unlevered cash flows of the asset are constant for time = 1 ... *n*, i.e., they are an annuity and (ii) debt principal is paid at maturity time = *n*. For these reasons the temporal subscript *t* is consistently deleted. The statement (see Appendix A) of the finite life WACC discount rate r_{WACC} is

$$IA_{WACC} = \frac{IA_e^L \times S^L}{\left(S^L + B\right)} + \frac{r_b \times B \times \left(1 - T_C\right)}{\left(S^L + B\right)} + \frac{IA_e^L \times P_n}{\left(S^L + B\right) \times \left(1 + r_e^L\right)^n} \quad , \tag{7}$$

where $IA_r = \frac{r}{\left(1 - \left(1 + r\right)^{-n}\right)}$ with $r = r_{WACC}$ or r_e^L; r_e^L represents the required rate of return on levered equity, r_b is the required rate of return on debt, and P_n (= *B*) is the principal paid at maturity. The translation of $IA_r \leftrightarrow r$ is straightforward. Beranek (1975, equation (1.21), p. 11), based on the same assumptions, is a cumbersome precursor of our model. Consider the special cases of the finite life WACC model. For the unlevered firm we get $r_{WACC} = r_e^U$ where r_e^U is the required rate of return on unlevered equity. This is the expected result. For a perpetual world $n = \infty$ and for a single period $n = 1$ we get

$$r_{WACC} = \frac{r_e^L \times S^L}{\left(S^L + B\right)} + \frac{r_b \times B \times \left(1 - T_C\right)}{\left(S^L + B\right)} \quad , \tag{8}$$

which is the textbook weighted average cost of capital. Clearly, this latter model is not appropriate in a finite life unless there are additional restrictive assumptions (Miles & Ezzell, 1980). This, in part, is an explanation for the discussions of repayment schedules in Miller

(2009a, 2009b), Bade (2009) and Pierru (2009a, 2009b).

In a practical context, the application of this finite life WACC formula requires an appropriate required rate of return on levered equity r_e^L. The textbook version is

$$r_e^L = r_e^U + \left(r_e^U - r_b\right) \times \frac{B}{S^L} \times \left(1 - T_C\right) \quad , \tag{9}$$

see also MM (1963, equation (12.c), p. 439). This model clearly applies to a perpetual world, but as we show later, it does not apply in a single period world. The corresponding model in Miles & Ezzell (1980, equation (22), p. 727) appears to be of limited utility in a practical sense since it is a function of the WACC. Using the assumptions adopted in the finite life WACC model, the corresponding finite life required rate of return on levered equity (see Appendix B) is determined via the non-linear equation

$$\begin{aligned} IA_e^L = {} & IA_e^U \times \frac{S^L}{\left(S^L + \frac{P_n}{\left(1 + r_e^L\right)^n}\right)} + \left(IA_e^U - IA_b\right) \times \frac{B}{\left(S^L + \frac{P_n}{\left(1 + r_e^L\right)^n}\right)} \times \left(1 - T_C\right) \\ & + \frac{P_n}{\left(S^L + \frac{P_n}{\left(1 + r_e^L\right)^n}\right) \times \left(1 + r_b\right)^n} \times \left[IA_e^U \times T_C + IA_b \times \left(1 - T_C\right)\right] \end{aligned} \quad , \tag{10}$$

which appears to be unwieldy. Notwithstanding, it is easy to programme into a spreadsheet, and then solve for r_e^L using an iterative procedure. When $n = \infty$, we get the textbook model (equation 9). When $n = 1$, we get

$$r_e^L = r_e^U + \left(r_e^U - r_b\right) \times \frac{B}{S^L} \times \left(1 - \frac{r_b}{\left(1 + r_b\right)} \times T_C\right) \quad , \tag{11}$$

which is popular in the literature. It is associated with the case when the leverage ratio is a constant (Fernandez, 2004, Table 2, p. 156; Arzac & Glosten. 2005, equation (31), p. 458). It is the specification of the required rate of return on levered equity for use in the textbook WACC under the Miles & Ezzell (1980) assumption of constant leverage.

Miller's (2009a, equation (23), p. 135) finite life WACC model, using our notation, is

$$IA_{WACC} = IA_e^L \times \frac{S^L}{S^L + B} + IA_b \times \frac{B}{S^L + B} \quad . \tag{12}$$

Miller (2009a, equation (24), p. 135) shows that this equation becomes

$$r_{WACC} = r_e^L \times \frac{S^L}{S^L + B} + r_b \times \frac{B}{S^L + B} \quad , \tag{13}$$

when $n = \infty$. A similar result is observed for a single period world, i.e., $n = 1$. This attests to the validity of Miller's (2009a) model. Equation (13) is the textbook WACC in a world where interest paid is not tax deductible, i.e., a MM (1958) world where $V^L = V^U$. In a similar world, our finite life model, i.e., equation (7) when $T_C = 0$, becomes

$$IA_{WACC} = \frac{IA_e^L \times S^L}{\left(S^L + B\right)} + \frac{r_b \times B}{\left(S^L + B\right)} + \frac{IA_e^L \times P_n}{\left(S^L + B\right) \times \left(1 + r_e^L\right)^n} \quad , \tag{14}$$

which is different from Miller (2009a). The explanation is based on differences in the debt repayment schedule. There will also be a similar effect with the required rate of return on levered equity. The implication is that there is a finite life WACC model for each type of debt repayment schedule. This issue of practicality in implementing the finite life WACC model as formulated above may not be a problem since there is a more straightforward way to estimate the finite life WACC.

4. Another View of the Finite Life WACC

Consider the conventional textbook WACC model (equation 8) and the textbook required rate of return of levered equity r_e^L (equation 9). As is well known, substitution for r_e^L into the WACC equation, followed by some rearrangement, ultimately gives

$$r_{WACC} = r_e^U \times \left(1 - \frac{B}{V^L} \times T_C\right) \quad , \tag{15}$$

which is the same as MM (1963, equation (31.c), p. 438). Massari, Roncaglio & Zanetti (2007, p. 159) show that this model eventuates in a world of perpetual growth under the Miles & Ezzell (1985) assumption that the tax saving for the first year is discounted at the cost of debt and the tax savings for the following years are discounted at the unlevered cost of equity. To their credit, Massari, Roncaglio & Zanetti (2007) question the degree that their assumptions are representative of the real world. They suggest that the APV method may be more appropriate.

We derive the equivalent finite life version of equation (15). The only assumption relates to a level series of uniform post-tax unlevered cash flows, for *n* periods, which are denoted by X^U. The derivation is silent on the debt repayment schedule. The value of the unlevered firm is

$$V^U = \frac{X^U}{1+r_e^U} + \frac{X^U}{\left(1+r_e^U\right)^2} + \cdots + \frac{X^U}{\left(1+r_e^U\right)^n} \tag{16}$$

and the value of the levered firm is

$$V^L = \frac{X^U}{1+r_{WACC}} + \frac{X^U}{\left(1+r_{WACC}\right)^2} + \cdots + \frac{X^U}{\left(1+r_{WACC}\right)^n} \quad . \tag{17}$$

Application of the annuity operator, followed by rearrangement gives $X^U = IA_e^U \times V^U$ and $X^U = IA_{WACC} \times V^L$. Then the elimination of X^U gives

$$IA_{WACC} = IA_e^U \times \frac{V^U}{V^L} \quad . \tag{18}$$

From the rearranged finite life MM (1963) model $V^U = V^L - PV\left(INT\right) \times T_C$ we get

$$IA_{WACC} = IA_e^U \times \left[1 - \frac{PV\left(INT\right) \times T_C}{V^L}\right] \quad . \tag{19}$$

This is a general statement of the finite life WACC model – it is independent of the debt repayment schedule.

For a MM (1963) perpetual world, where $PV\left(INT\right) = B$, and when the interest tax shield is discounted at the cost of debt, we get the textbook equation (15). For a single period world, where $IA_r = 1 + r$ and $PV\left(INT\right) \times T_C = \frac{r_b}{\left(1+r_b\right)} \times B \times T_C$, we get

$$r_{WACC} = r_e^U - \left(1 + r_e^U\right) \times \frac{B}{V^L} \times \frac{r_b}{\left(1+r_b\right)} \times T_C \ , \tag{20}$$

which is Miles & Ezzell (1980, equation (20), p. 726).

There is also the issue of how to determine the required rate of return on unlevered equity r_e^U. This is needed to calculate the IA_e^U element in the finite life WACC model (equation 19). It is also used to determine the unlevered net present value NPV_0^U in the APV method (equation 3). For convenience, let us focus on a levered firm, the owner of a single asset, whose equity is listed on the stock exchange. Conventionally, a two-step process is adopted. The first step is to determine the beta of the equity of the levered firm β_e^L -- the Market Model is used for this purpose. The second step is to 'strip away' the effects of leverage to achieve the beta of the equity of the unlevered firm β_e^U -- the Hamada (1972) model is used for this purpose. This textbook formula,

$$\beta_e^L = \beta_e^U \times \left[1 + \frac{B}{S^L} \times (1 - T_C)\right] \quad , \tag{21}$$

is based on the MM (1963) perpetual debt model (see equation 1) and hence may not apply in a finite world. The required rate of return on unlevered equity can be achieved in the normal manner by applying β_e^U, finessed by deduction from the equation, to the Capital Asset Pricing Model.

Our derivation of a finite life Hamada (1972) model starts with the finite life MM (1963) model which is written as

$$S^L + B \equiv V^L = V^U + PV[INT_t] \times T_C \quad . \tag{2}$$

Noting that betas are additive when weighted by market value, we get

$$\beta_e^L \times \frac{S^L}{V^L} + \beta_b \times \frac{B}{V^L} = \beta_e^U \times \frac{S^U}{S^U + PV(INT_t) \times T_C} + \beta_b \times \frac{B}{S^U + PV(INT_t) \times T_C} \quad , \tag{22}$$

where $S^U \equiv V^U$. Assuming that debt is essentially risk free, i.e., $\beta_b = 0$, followed by simple rearrangement, gives

$$\beta_e^L = \beta_e^U \times \frac{S^U}{S^U + PV(INT_t) \times T_C} \times \frac{V^L}{S^L} \quad . \tag{23}$$

Noting from the finite life MM (1963) that $V^L = S^U + PV(INT_t) \times T_C$, we get

$$\beta_e^L = \beta_e^U \times \frac{S^U}{S^L} \tag{24}$$

(see Hamada, 1972, equation (4), p. 439). Further recourse to the finite life MM (1963) model, i.e., $S^U = S^L + B - PV(INT_t) \times T_C$ gives, after rearrangement,

$$\beta_e^L = \beta_e^U \times \left[1 + \frac{B}{S^L} - \frac{PV(INT_t) \times T_C}{S^L}\right] \quad . \tag{25}$$

This is the finite life Hamada (1972) model. In a perpetual world and when the interest tax relief is discounted at the cost of debt, $PV(INT_t) \times T_C = B \times T_C$, we achieve the textbook

Hamada (1972) model (equation 21).

For a single period world, where $PV\left(INT_t\right)\times T_C = \frac{r_b}{\left(1+r_b\right)}\times B\times T_C$, we achieve the Miles & Ezzell (1985, equation (27), p. 1491) model

$$\beta_e^L = \beta_e^U \times\left[1+\frac{B}{S^L}-\frac{B}{S^L}\times\frac{r_b}{\left(1+r_b\right)}\times T_C\right] \quad . \tag{26}$$

Miles & Ezzell (1985, Abstract, p. 1485) claim that this model is the Hamada (1972) equivalent under the assumptions of: (i) a perpetual world, (ii) a constant leverage ratio, (iii) the tax saving for the first year is discounted at the cost of debt and (iv) the tax savings for the following years are discounted at the unlevered cost of equity. Arzac & Glosten (2005, p. 458), employing the same assumptions, present an identical model. Equation (26) can also be obtained from the one-period required rate of return on levered equity model (equation 11) by the application of the CAPM followed by simple rearrangement.

5. Concluding Remarks

The very essence of any WACC model is that the present value of the benefits of the tax relief on interest paid -- which are in dollar terms -- are acknowledged by a reduction in the discount rate. This aspect is the underlying cause of the debate raised in response to Miller (2009a). Economic logic suggests that the interest tax benefit should be matched to the discount rate for the period in which the benefit occurs. There will be a uniform interest tax benefit each year if a constant leverage is assumed (Miles & Ezzell, 1980). Thus a constant WACC is indicated -- see Bade (2009, Table 2, p. 1479) or Pierru (2009a, Table 1, p. 1221). However, now consider debt redeemed by level annuity repayments. The periodic interest tax benefit will be larger at the start of the project compared to the end of the project. The economic matching principle suggests that a uniform WACC is contraindicated. A different WACC for every year is warranted -- the WACC should increase over time -- see Bade (2009, Table 1, p. 1478) or Pierru (2009a, Table 2, p. 1222). This aspect can be addressed by calculating the WACC on a year-by-year basis using expecations of the way the leverage ratio will vary over time. This is a practical way to bypass the sometimes unrealistic temporal assumption of constant leverage required to make the WACC a constant.

Our analysis, as does the analysis of others, raises the issue of whether it is better, in a conceptual sense, to account for the interest tax benefit as a dollar value or to account for it as an adjustment to the discount rate. Is the Adjusted Present Value Model superior to the WACC model? The APV model is straightforward. The potential problems in the determination of: (i) the required rate of return on unlevered equity r_e^U (ii) the present value of the tax shield $PV(INT_t)\times T_C$ (see Fernandez (2004, Table 1, p. 156) for a survey of the literature) are common to the finite life WACC method and the APV method. The application of Occam's Razor (Ennis, 2009) infers that the APV method is preferred to the WACC method. The WACC is a special case of the APV (Miles & Ezzell, 1980) and therefore it is based on additional assumptions. However, although is possible that the textbook "weighted average cost of capital is not quite right" (Miller, 2009a), can one be confident that the APV model is "quite right"?

References

Arzac, E. and Glosten, L. (2005). A reconsideration of tax shield valuation. *European Financial Management*, 11, 453-461.

Bade, B. (2009). Comment on "The weighted average cost of capital is not quite right". *Quarterly Review of Economics and Finance*, 49, 1476–1480.

Beranek, W. (1975). The cost of capital, capital budgeting, and the maximization of shareholder wealth. *Journal of Financial and Quantitative Analysis*, 10, 1-20.

Ennis, R. M. (2009). Parsimonious asset allocation. *Financial Analysts Journal*, 65, 6-10.

Fernandez, P. (2004). The value of tax shields is NOT equal to the present value of tax shields. *Journal of Financial Economics*, 73, 145-165.

Hamada, R. S. (1972). The effect of the firm's capital structure on the systematic risk of common stocks. *Journal of Finance*, 27, 435-452.

Harris, R. S. and Pringle, J. J. (1985). Risk-adjusted Discount Rates – Extensions from the Average-risk Case. *The Journal of Financial Research*, 8, 237-244.

Massari, M., Roncaglio, F. and Zanetti, L. (2007). On the equivalence between the APV and the wacc approach in a growing leveraged firm. *European Financial Management*, 14, 152-162.

Miles, J. and Ezzell, J. (1980). The weighted average cost of capital, perfect capital markets, and project life: A clarification. *Journal of Financial and Quantitative Analysis*, 15, 719-730.

Miles, J. and Ezzell, J. (1985). Reformulating tax shield valuation: A note. *Journal of Finance*, 40, 1485-1492.

Miller, R. A. (2009a). The weighted average cost of capital is not quite right. *Quarterly Review of Economics and Finance*, 49, 128–138.

Miller, R. A. (2009b). The weighted average cost of capital is not quite right: Reply to M. Pierru. *Quarterly Review of Economics and Finance*, 49, 1213-1218.

Modigliani, F. and Miller, M. (1958). The cost of capital, corporate finance and the theory of investment. *American Economic Review*, 48, 261-297.

Modigliani, F. and Miller, M. (1963). Corporate income taxes and the cost of capital: A correction. *American Economic Review,* 53, 433-443.

Myers, S. (1974). Interactions of corporate financing and investment decision implications for capital budgeting. *Journal of Finance*, 29, 1-25.

Pierru, A. (2009a). The weighted average cost of capital is not quite right: A comment. *Quarterly Review of Economics and Finance*, 49, 1219–1223.

Pierru, A. (2009b). The weighted average cost of capital is not quite right: A rejoinder. *Quarterly Review of Economics and Finance*, 49, 1481–1484.

Pierru, A. and Babusiaux, D. (2010). WACC and free cash flows: A simple adjustment for capitalized interest costs. *Quarterly Review of Economics and Finance*, 50, 240–243.

Ross, S. A., Westerfield, R. W. and Jaffe, J. (2010). *Corporate Finance*, Ninth edition, McGraw-Hill/Irwin.

Appendix A

Derivation of Finite Life WACC model

The explicit assumptions are: (i) the unlevered cash flows of the asset CF^U are constant for time = 1 ... *n*, i.e., they are an annuity and (ii) debt receives a constant stream of interest payments and debt principal is paid at maturity time = *n* and (iii) interest paid is tax deductible.

The application of the WACC method to determine the net present value NPV_0 of the project gives

$$NPV_0 = \frac{CF^U \times (1-T_C)}{1+r_{WACC}} + \ldots + \frac{CF^U \times (1-T_C)}{(1+r_{WACC})^n} - Cost_0 \quad , \tag{A.1}$$

where r_{WACC} is the WACC discount rate and $Cost_0$ is the cost of the investment, assumed to occur at time = 0, consisting of the initial cash contribution S_0 by equity and the cash contribution by debt *B*, that is, $Cost_0 = S_0 + B$. The first step is to estimate $CF^U \times (1-T_C)$ from equation (A.1). The second step is to obtain an independent estimate of $CF^U \times (1-T_C)$ by the use of the Flow-to-Equity model. The elimination of $CF^U \times (1-T_C)$ leads to the finite life WACC model.

In the first step, A_{WACC} is used to represent the annuity present value function for r_{WACC}, that is, $A_{WACC} = \frac{1}{r_{WACC}} \times \left(1-(1+r_{WACC})^{-n}\right)$. Thus equation (A.1) is written as

$$NPV_0 = A_{WACC} \times CF^U \times (1-T_C) - Cost_0 \quad , \tag{A.2}$$

and solving for $CF^U \times (1-T_C)$ gives

$$CF^U \times (1-T_C) = \frac{NPV_0 + Cost_0}{A_{WACC}} = \frac{S^L + B}{A_{WACC}} \quad , \tag{A.3}$$

where $NPV_0 + Cost_0 = S^L + B$, that is to say, the net present value, which accrues to equity, i.e., $S^L = S_0 + NPV_0$, is immediately reflected in the market value of the levered firm.

The second step uses the Flow-to-Equity net present value model -- this is where the cash flows to equity are discounted at the required rate of return on levered equity r_e^L, thus

$$S^L = \frac{\left(CF^U - INT\right)\times\left(1-T_C\right)}{1+r_e^L} + \cdots + \frac{\left(CF^U - INT\right)\times\left(1-T_C\right) - P_n}{\left(1+r_e^L\right)^n} \quad , \tag{A.4}$$

where principal is paid at maturity. Separating the terms in the numerators and the application of the annuity present value operator $A_e^L = \frac{1}{r_e^L}\times\left(1-\left(1+r_e^L\right)^{-n}\right)$ gives

$$S^L = A_e^L \times CF^U \times \left(1-T_C\right) - A_e^L \times INT \times \left(1-T_C\right) - \frac{P_n}{\left(1+r_e^L\right)^n} \quad . \tag{A.5}$$

Solving for $CF^U \times \left(1-T_C\right)$ gives

$$CF^U \times \left(1-T_C\right) = \frac{S^L + A_e^L \times r_b \times B \times \left(1-T_C\right) + P_n / \left(1+r_e^L\right)^n}{A_e^L} \quad , \tag{A.6}$$

where $INT = r_b \times B$ since principal is paid at maturity.

The elimination of $CF^U \times \left(1-T_C\right)$ from equation (A.3) and equation (A.6) gives

$$\frac{S^L + B}{A_{WACC}} = \frac{S^L + A_e^L \times r_b \times B \times \left(1-T_C\right) + P_n / \left(1+r_e^L\right)^n}{A_e^L} \quad . \tag{A.7}$$

Dividing both sides by $\left(S^L + B\right)$ and using the inverses of the annuity present value operators $IA_{WACC} = \frac{1}{A_{WACC}} = \frac{r_{WACC}}{\left(1-\left(1+r_{WACC}\right)^{-n}\right)}$ and $IA_e^L = \frac{1}{A_e^L} = \frac{r_e^L}{\left(1-\left(1+r_e^L\right)^{-n}\right)}$ and separating the terms on the right hand side of equation (A.7), we get

$$IA_{WACC} = \frac{IA_e^L \times S^L}{\left(S^L + B\right)} + \frac{r_b \times B \times \left(1-T_C\right)}{\left(S^L + B\right)} + \frac{IA_e^L \times P_n}{\left(S^L + B\right)\times\left(1+r_e^L\right)^n} \quad . \tag{A.8}$$

This is a statement of the finite life WACC discount rate for a maturity of *n* years with debt principal repaid at maturity at time = *n*.

Appendix B

Derivation of the required rate of return on levered equity

As discussed before, the Flow-to-Equity model discounts the net cash flows to equity by the r_e^L. The finite model is

$$S^L = \frac{\left(CF^U - INT\right)\times\left(1-T_C\right)}{1+r_e^L} + \cdots + \frac{\left(CF^U - INT\right)\times\left(1-T_C\right) - P_n}{\left(1+r_e^L\right)^n} \quad . \tag{A.4}$$

Using the annuity present value operator A_e^L the equation can be written as

$$S^L = A_e^L \times \left(CF^U - INT\right) \times \left(1 - T_C\right) - \frac{P_n}{\left(1 + r_e^L\right)^n} \quad , \tag{B.1}$$

which can be arranged to

$$S^L + \frac{P_n}{\left(1 + r_e^L\right)^n} = A_e^L \times \left(CF^U - INT\right) \times \left(1 - T_C\right) \quad . \tag{B.2}$$

Splitting the terms on the right hand side and applying the inverse of the annuity present value operator IA_e^L gives

$$IA_e^L \times \left(S^L + \frac{P_n}{\left(1 + r_e^L\right)^n}\right) = CF^U \times \left(1 - T_C\right) - INT \times \left(1 - T_C\right) \quad . \tag{B.3}$$

The analytical procedure that follows is to convert the term $CF^U \times \left(1 - T_C\right)$ and the term $INT \times \left(1 - T_C\right)$ into the product of a value and an interest rate.

Consider the first term $CF^U \times \left(1 - T_C\right)$. The definition of the value of unlevered equity

$$S^U = \frac{CF^U \times \left(1 - T_C\right)}{1 + r_e^U} + \cdots + \frac{CF^U \times \left(1 - T_C\right)}{\left(1 + r_e^U\right)^n} = A_e^U \times CF^U \times \left(1 - T_C\right) \quad , \tag{B.4}$$

where A_e^U is the the annuity present value function for unlevered equity. Rearrangement and using $IA_e^U = 1 / A_e^U$ gives

$$CF^U \times \left(1 - T_C\right) = S^U \times IA_e^U \quad . \tag{B.5}$$

The finite life MM (63) valuation model

$$V^L = V^U + PV\left(INT\right) \times T_C \tag{B.6}$$

gives, after noting $S^U = V^U$,

$$S^U = S^L + B - PV\left(INT\right) \times T_C \quad . \tag{B.7}$$

Noting

$$B \equiv PV\left(INT\right) + P_n / \left(1 + r_b\right)^n \text{ giving } PV\left(INT\right) = B - P_n / \left(1 + r_b\right)^n \tag{B.8}$$

we get

$$S^U = S^L + B - \left(B - P_n / \left(1 + r_b\right)^n\right) \times T_C = S^L + B \times \left(1 - T_C\right) + \frac{P_n}{\left(1 + r_b\right)^n} \times T_C \quad . \tag{B.9}$$

Substitution into equation (B.5) gives

$$CF^U \times (1-T_C) = IA_e^U \times \left(S^L + B \times (1-T_C) + \frac{P_n}{(1+r_b)^n} \times T_C \right) \quad . \tag{B.10}$$

Now consider the second term $INT \times (1-T_C)$. The present value of the series of interest payments *INT* is given by $PV(INT) = A_b \times INT$ where A_b is the annuity present value operator of r_b. So $INT = IA_b \times PV(INT)$. Using $B \equiv PV(INT) + P_n/(1+r_b)^n$ we get $PV(INT) = B - P_n/(1+r_b)^n$. Thus

$$INT = IA_b \times \left(B - P_n/(1+r_b)^n\right) \quad , \tag{B.11}$$

where $IA_b = 1/A_b$. Thus adding the $(1-T_C)$ term gives

$$INT \times (1-T_C) = IA_b \times \left(B - P_n/(1+r_b)^n\right) \times (1-T_C) \quad . \tag{B.12}$$

Going back to equation (B.3), namely,

$$IA_e^L \times \left(S^L + \frac{P_n}{(1+r_e^L)^n} \right) = CF^U \times (1-T_C) - INT \times (1-T_C) \tag{B.3}$$

and making the substitutions from equations (B.10) and (B.12) we get

$$\begin{aligned} IA_e^L \times \left(S^L + \frac{P_n}{(1+r_e^L)^n} \right) &= IA_e^U \times \left(S^L + B \times (1-T_C) + \frac{P_n}{(1+r_b)^n} \times T_C \right) \\ &\quad - IA_b \times \left(B - P_n/(1+r_b)^n\right) \times (1-T_C) \end{aligned} \tag{B.13}$$

which expands to

$$\begin{aligned} IA_e^L \times \left(S^L + \frac{P_n}{(1+r_e^L)^n} \right) &= IA_e^U \times S^L \\ &\quad + IA_e^U \times B \times (1-T_C) \\ &\quad + IA_e^U \times \frac{P_n}{(1+r_b)^n} \times T_C \\ &\quad - IA_b \times B \times (1-T_C) \\ &\quad + IA_b \times \frac{P_n}{(1+r_b)^n} \times (1-T_C) \end{aligned} \quad . \tag{B.14}$$

Collection of like terms gives

$$IA_e^L \times \left(S^L + \frac{P_n}{\left(1+r_e^L\right)^n} \right) = IA_e^U \times S^L + \left(IA_e^U - IA_b\right) \times B \times \left(1 - T_C\right)$$
$$+ \frac{P_n}{\left(1+r_b\right)^n} \times \left[IA_e^U \times T_C + IA_b \times \left(1 - T_C\right) \right] , \qquad \text{(B.15)}$$

which rearranges to

$$IA_e^L = IA_e^U \times \frac{S^L}{\left(S^L + \frac{P_n}{\left(1+r_e^L\right)^n} \right)} + \left(IA_e^U - IA_b\right) \times \frac{B}{\left(S^L + \frac{P_n}{\left(1+r_e^L\right)^n} \right)} \times \left(1 - T_C\right)$$
$$+ \frac{P_n}{\left(S^L + \frac{P_n}{\left(1+r_e^L\right)^n} \right) \times \left(1+r_b\right)^n} \times \left[IA_e^U \times T_C + IA_b \times \left(1 - T_C\right) \right] . \qquad \text{(B.16)}$$

This is the statement for the finite life required rate of return on levered equity when debt principal is paid at maturity.