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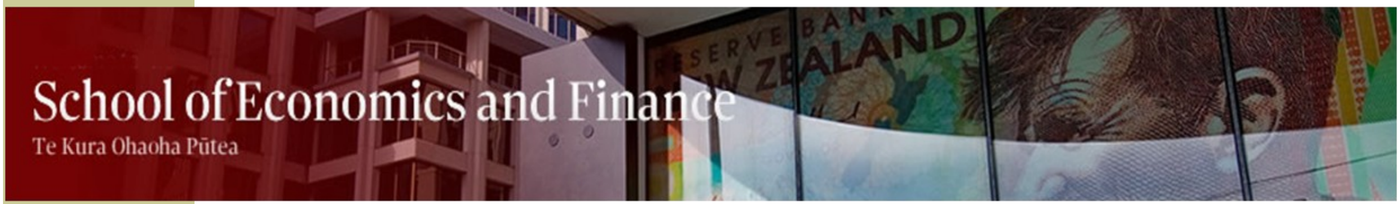

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Approximations to viability kernels for sustainable macroeconomic policies

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February 17, 2011

Abstract

Maintaining an open economy within certain bounds on inflation, output gap and exchange rate can help sustainable economic development. Macroeconomics proposes monetary-policy models that describe evolution of the above quantities. We use one such model, constituted by a four-metastate one-control system, to compute viability kernel approximations that one can use to assist the central bank to establish "sustainable" policies. We propose a simple heuristic algorithm that leads to kernel approximations for this and similar models.

Keywords: monetary policy; viability kernel, MATLAB
JEL:C6, C61, C69
MSC: 34H05, 34K35, 49J15, 49L25, 91B02, 91B62, 93C15

1 Introduction

This paper[1] is concerned with inflation-targeting in an economy. Rather than following the traditional approach (see *e.g.*, [15], [27], [33]), which consists of

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[1]This paper builds on and extends research documented in [19] and [22].

optimisation of a loss function, we pursue an alternative approach[2] based on viability theory (see *e.g.*, [2], [1] or [3]).

Given a closed set with some given normative constraints and initial conditions, viability theory attempts to determine a control strategy such that the system will not violate those constraints. We call a closed set, with boundary and interior points for which there exist controls such that the system never leaves the constraint set, a *viability domain.* The largest such set is called a *viability kernel.* A system at any point exterior to the viability kernel is hence unstable and doomed to eventually leave the constraint set.

As observed in [24], there are several advantages to a viable solution over an optimal one. These include robustness to shocks and parameter uncertainty and also enhanced credibility of the central bank's decisions. We will comment on these advantages later in the paper.

Here, we highlight some links between viability theory, bounded rationality and sustainability. Briefly, the existence and importance of these links explicates an economic interest in viability theory[3].

Herbert A. Simon, 1978 Economics Nobel Prize laureate, argued that there are *bounds* on economic agents' "rationality" and that economists really need "satisficing", (his neologism, see [32]) rather than optimising solutions. We share Simon's view and believe that some economic agents may not seek unique optimal solutions. Take for example the central bank governor's task, in a country where the allowable inflation band has been legislated, or a national park director who is responsible for biodiversity of the fauna, or an international body seeking multi-country adherence to some standards. Each of them will strive to satisfy several goals, many of them consisting of ensuring that the key outcomes (*e.g.*, inflation or the number of bears or the noxious substance amount) remain within some normative bounds. The bounds might result from some felicity function optimisation, but the governor (or park director, or the international body) will perceive them as exogenously specified. We think that an economic theory that follows the Simon prescription may bring modelling closer to how these people actually behave.

[2] We will emulate the approach to establish a "satisficing" monetary policy started in [24] and continued in [18], [21], and [22] .

[3] Viability theory has been successfully applied to environmental economics problems see [6], [25], [12] and [26]; for applications to financial analysis see [30] and the references provided there. Along with [24], [8], [9], [18], [21], [11] and [10] deal with viable solutions to macroeconomic problems; see [20] for a microeconomic problem solution.

We contend[4] that viability theory rigorously captures the essence of *satisficing*. We follow this conjecture and propose a method based on viability theory to produce satisficing solutions to a monetary policy problem. Other problems concerning biodiversity protection, international adherence to standards or sustainable development can be solved using the *same* method. The connection with the latter is through a *sustainability screw* (see [14]), a looping evolutionary trajectory that represents a sustainable development scenario as long as it remains within a constraint set.

As said, solving a viability problem requires computation of the viability kernel. This is difficult and the level of difficulty increases with the problem's dimensionality. Using algorithms (*e.g.*, [7], [31]) or heuristics, the papers above cited in footnote 3 provide examples of viability kernels in two and three dimensional state spaces. Our problem is in four dimensions. In [19] a method based on some optimal control results from [16] was utilised for this same problem, whereby a viability kernel was approximated by those state space locations for which the value function realisations of an *auxiliary* cost-minimising optimal control problem were small. In this paper, we use the same idea and propose an algorithm that detects state space points from which the available controls can *stabilise* the system at a steady state.[5]

What follows is a brief outline of what this paper contains. In Section 2, we describe a monetary policy model for which we will seek a satisficing solution defined in Section 3. A base for a solution that amounts to the viability domain's computation is provided in Section 5. Then, in Section 6 we present the approximations to the viability kernels for the satisficing policies of the central bank. The concluding remarks summarise our findings.

2 A monetary policy model

2.1 System's dynamics

We will now introduce a monetary policy model suitable for analysis through viability theory, following [24] and [22].

Typically (see *e.g.*, [28] or [33]), a central bank aims to achieve the maintenance of a few key macroeconomic variables within some bounds. Usually (see, *e.g.*, [34]), the bank realises its multiple targets using *optimising* so-

[4]We concord with Filar's conjecture [13].

[5]These points are hence members of a viability domain.

lutions that result from minimisation of the bank's loss function. The loss function may include, amongst other things, penalties for violating an allowable inflation band, for unemployment and, frequently, for non-smooth interest rate adjustments. The solution, which minimises the loss function, is unique for a given selection of the loss function parameters (including discount rate) in that it does not allow for alternative strategies — no other strategy will be considered optimal, given the parameters. Our intention is to obtain a *set* of satisficing strategies that allow the bank to keep the variables of interest in a constrained set.

Suppose a central bank uses a nominal short-term interest rate as an instrument to control inflation and, to a lesser extent, the output gap[6] and to an even lesser extent, the real exchange rate[7]. A model that relates these variables may look as follows:

$$\begin{aligned} y_t &= \delta_1 y_{t-h} - \delta_2\Big(i_{t-h} - E_{t-h}\pi_t\Big) + \delta_3 q_{t-h} + \varepsilon^y_{t-h} && (1) \\ \pi_t &= \phi_0 E_t\pi_{t+h} + (1-\phi_0)\pi_{t-h} + \phi_1\Big(y_t + y_{t-h}\Big) + && (2) \\ &\quad + \mu\Big((1-\phi_0)\Delta q_t - \phi_0 E_t\Delta q_{t+h}\Big) + \varepsilon^\pi_t && \\ E_t\Delta q_{t+h} &= \Big(i_t - E_t\pi_{t+h}\Big) + \varepsilon^q_t\,. && (3) \end{aligned}$$

This model is a version of that described in [4] and [5], which can also be viewed as an extension of [34] and [33], and which gave rise to the continuous time version of the model in [24]. Also see [23] and [18] where the same model was experimented with by adding a foreign "nuisance" agent.

In the current version of the model we assume that dominant factor for determining inflation is inflation *persistence* and set $\phi_0 = 0$ (we will then use ϕ instead of ϕ_1).

In (2) and (3), Δ is the first-difference operator $\Delta x_t = x_t - x_{t-h}$ where x_t can be any of the above variables, h is the amount of time between realisations

[6]We adopt the common meaning of *output gap* *i.e.*, output gap is the log deviation of actual output from the normal or potential level.

[7]The real exchange rate q_t is defined as the *log* ratio of *nominal exchange rate* $\times$ *foreign price index* to *domestic price index* and can be viewed as an aggregate measure of strength of a country's currency. If the currency weakens, then q_t increases. *I.e.*, larger (positive) q_t means real depreciation (more local currency units are needed for one unit of the foreign currency) so that domestic goods become relatively cheaper. Symmetrically, if q diminishes, less local currency units is needed for a unit of the foreign currency, which means that the foreign goods become cheaper.

of x, and $E_{t-h}x_t$ is the variable's expected value formed at time $t-h$. This and assuming that x_t is time-t realisation of a continuous-time stochastic process $x(t)$, $t \in \Theta$ and that agents do not expect shocks for some h (so, the sample paths of $x(t)$ are piece-wise "smooth"), allows us to approximate expectation formation through extrapolation (or as simple learning processes; compare [17]) and receive (for details, see [22][8]):

$$\begin{aligned} \dot{y} &= ay - d_2(i-\pi) + d_3 q && (4)\\ \dot{\pi} &= 2py && (5)\\ \dot{q} &= (i-\pi)\,. && (6) \end{aligned}$$

where $y(t), \pi(t), q(t)$, $t \in (0,\infty)$ are expected values of output gap, inflation and exchange rate and constitute the state vector while $i(t)$ is annualised nominal short-term interest rate. The latter is used by the central bank to control the economy. Parameters a, d_2 and p are positive. The coefficient on exchange rate d_3 is positive in the small open economy model. However, in the closed economy model $d_3 = 0$; consequently, equation (6) is dropped for a closed economy.

The obtained model (4)-(6) tells us that the expected output gap (see (4)) constitutes a "sticky" process driven by real interest rate (*i.e.*, by the term $(i-\pi)$). Moreover, the exchange rate affects competitiveness of domestic goods in the world market, so it also affects the output gap changes: if the domestic currency appreciates (*i.e.*, $q(t)$ diminishes, $d_3 > 0$) then the output-gap growth slows down and may become negative.

The expected speed of inflation (see (5)) changes proportionally to the expected output gap *doubled*. Interestingly, in this model, the short-term (continuous-time) inflation rate expectation does not depend on the exchange rate differential (as in the discrete time model, see [4]), because the latter tends to zero for short periods.

Equation (6) captures the process of currency adjustment to real interest rate. This is a continuous time version of the classical *uncovered interest parity condition*. The condition implies the following mechanism: if the domestic real interest rate increases then bonds will earn more in local currency. There will also be increased demand from abroad for the bonds and the currency will "momentarily" appreciate. Then, the currency will depreciate so

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[8] In essence, our model is a continuous-time version of that described in [4] and [5]; see [22] for derivation. Also see [23] and [18] where we have experimented with a model based on [34] and [33].

that all countries' bonds yield the same return. We notice that equation (6) bypasses the surge in the exchange rate and describes the domestic agents' expectations guided solely through the uncovered interest parity condition. The surge can however be allowed for through a set of new initial conditions imposed on (4)-(6), after a "shock" in the nominal interest rate has been noticed.

We will use the parameter values reported by [4] and [5], calibrated on UK data. In result (see [22]), the macroeconomic model that we will analyse is:

$$\begin{aligned} \dot{y} &= -0.2y - 0.5(i - \pi) + 0.2q && (7) \\ \dot{\pi} &= 0.4y && (8) \\ \dot{q} &= (i - \pi)\,. && (9) \end{aligned}$$

3 The kernel problem

Usually there is little doubt as to what the *politically* desirable inflation bounds are. For example, in New Zealand, the (annualised) inflation band has been legislated to be $[.01,\ .03]$.[9] Recognising that the output-gap is a second order concern for many inflation-targeting central banks, we posit a wide interval for $y(t) \in [-.04,\ .04]$.

There is little agreement as to an *ideal* range for the real exchange rate. We assume a rather wide interval of acceptability with $q(t) \in [-.1,\ .1]$. So, our constraint set is

$$\begin{aligned} K \quad &\equiv \quad \{(y(t), \pi(t), q(t)) \,:\, -.04 \le y(t) \le .04, \;\; .01 \le \pi(t) \le .03 \\ &\text{and} \quad -.1 \le q(t) \le .1\}\,, \end{aligned} \tag{10}$$

as shown in Figure 1.

As with the desired size of inflation, output gap and real exchange rate, the instrument set composition also depends on political decisions. We assume that $i(t) \in [0,\ .07]$.

The lower bound on $i(t)$ is obvious; the upper bound seems "historically" justified as it was only infrequently violated in countries like the US or Japan. We notice that official interest rates have certainly been higher recently in

[9] More precisely, the Reserve Bank of New Zealand aims to deliver inflation that is between one and three percent on average over the medium-term.

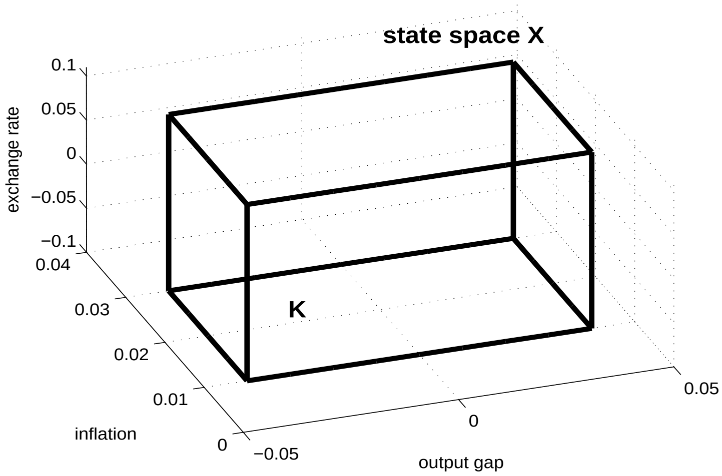

Figure 1: Constraint set $K \in \mathcal{R}^3$.

countries like New Zealand. However, it is important to remember that these were at times when inflation was significantly in excess of the upper bound of the target band, and when the output gap was large and positive. As we are only interested in viable solutions, inflation and the output gap will *always* remain inside their respective bounds. In this light, a 7 percent cap on interest rates appears to be a reasonable constraint.

Central banks may be even more concerned about managing interest rate volatility as compared with constraining their level. This concern is usually modelled by adding $w(i(t) - i(t-h))^2$, $w > 0$ to the loss function. In continuous time, limiting the interest rate "velocity" $\dfrac{di}{dt}$ will produce a smooth time profile for $i(t)$. Bearing in mind that the central bank's announcements are usually made monthly and that the typical change, when made, is about $\dfrac{1}{4}$% per announcement, the instrument set should allow for some tolerance and will be here defined as

$$\mathfrak{I} \equiv \left\{ i \,:\, i(t) \in [0,\, 0.07], \quad \text{and} \quad \frac{di}{dt} \in U = [-0.01,\, 0.01] \right\} \tag{11}$$

i.e., the interest rate can drop, or increase, between 0 and 1% per quarter.

The constraints on our instrument set, as specified by $\mathfrak{I}$, thus extend our constraint set, $K \in \mathcal{R}^3$ into a four-dimensional "metastate" constraint set,

$K \times [0,\, 0.07] \in \mathcal{R}^4$, which we are able to influence by altering the velocity at which the fourth dimension changes. In this conception of our viability problem, our control is $u = \dfrac{di}{dt} \in U = [-0.01,\, 0.01]$, the rate of change of interest, rather than the interest rate itself.

Hence, the dynamic system that links the output gap, inflation, exchange rate, interest rate and the change in interest rate now looks as follows:

$$\frac{dy}{dt} = -0.2y - 0.5(i - \pi) + 0.2q \tag{12}$$

$$\frac{d\pi}{dt} = 0.4y \tag{13}$$

$$\frac{dq}{dt} = (i - \pi) \tag{14}$$

$$\frac{di}{dt} = u \in U\,. \tag{15}$$

The above system of equations and inequalities define the dynamics of our open economy.

Several basic facts about the dynamics of monetary policy are apparent from Figure 2, which presents a projection $K \times [0,\, 0.07]$ into the space (*output gap, inflation*).

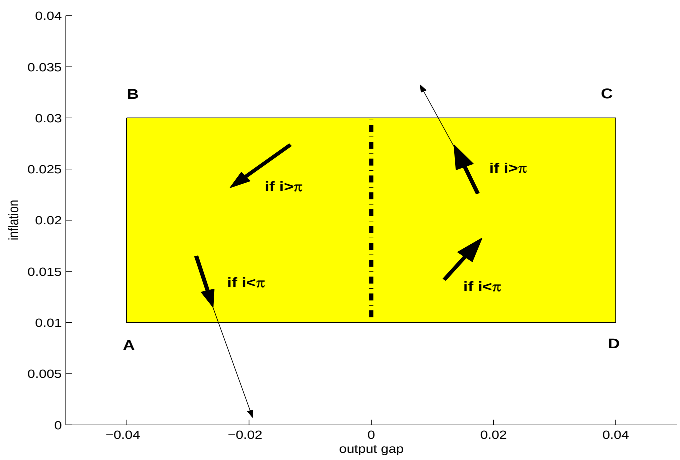

Figure 2: Approximated evolution of the economy on plane ${}^{p}X$.

Suppose that $q(t)$ is close to zero (thus exerting little to no influence on the system), and consider corner **C**. The output gap is positive and inflation is high. In such a scenario, any increase in the interest rate i must happen early enough to ensure that the upper bound on inflation is not violated. This timing imperative occurs because we require interest rates to move smoothly (see (11)). This means that the interest rate *adjustment speed* is constrained and any sudden hike in i is impossible. A central bank's natural concern therefore is for determining the collection of state space points from where the instruments available in $\mathfrak{I}$ are *sufficient* to prevent the system from evolving in such a way that it leaves the metastate constraint $K \times [0,\, 0.07]$. Those points in $K \times [0,\, 0.07]$, for which the instruments in $\mathfrak{I}$ are indeed sufficient should thus be in our viability kernel.

A different problem occurs if the output gap is negative (as is the case in the vicinity of corner $\mathbf{A}$). To avoid a *liquidity trap*[10] the economy must evolve such that negative output gap and low inflation states are avoided. So, in the vicinity of $\mathbf{A}$, monetary policy must be relaxed "early enough" so as to remain within $K \times [0,\, 0.07]$. That is, the interest rate must be lowered before the system's inertia leads to the lower limit on inflation being breached. If the bank does not start lowering nominal interest rate "sufficiently early" *i.e.*, when inflation is already close to the boundary (here, 1%) then the real interest rate control will be close to zero and ineffective at stimulating growth. Consequently, the economy will further drift toward zero inflation with a negative output gap. The points in $K \times [0,\, 0.07]$ where the instruments in $\mathfrak{I}$ are sufficient to avoid this scenario should thus be in our viability kernel as well.

Formally, let us call the system's dynamics $\Psi(x, u)$, which is the collective vector of the right hand sides of (12)-(15) where $x \equiv [y, \pi, q, i]' \in K \times [0,\, 0.07]$, and $u \in U$. That is, for any state-space vector, x, a solution to $\dot{x} = \Psi(x, u)$ gives the trajectory of the system from that point, when control u is applied. Then let $\mathcal{V}_{\cdot}([\dots])$ denote the *viability kernel* that satisfies the following definition:

Definition 1 *The* ***viability kernel*** *of the constraint set* $K \times [0,\, 0.07]$ *for the control set* U *and system's dynamics* $\Psi(x, u)$ *is the set of initial conditions*

[10] In a liquidity trap an economy remains in an area where the output gap is negative and inflation is close to zero (positive or negative); see [28] for an analysis of a liquidity trap problem through an established method. See [29] for an analysis of a liquidity trap problem in state space.

$x_0 \in K \times [0,\, 0.07]$, *denoted as* $\mathcal{V}_\Psi(K \times [0,\, 0.07])$ *and defined as follows:*

$$\begin{gathered}\mathcal{V}_\Psi(K \times [0,\, 0.07]) \equiv \\ \{x_0 \in K \times [0,\, 0.07] : \exists x(t) \text{ solution to } \dot{x} = \Psi(x(t), u) \text{ with } x(0) = x_0 \text{ s.t.} \\ x(t) \in K \times [0,\, 0.07], \forall t\}\,,\end{gathered} \tag{16}$$

The problem that we want to solve therefore is to establish viability kernel $\mathcal{V}_\Psi(K \times [0,\, 0.07]) \subset K \times [0,\, 0.07]$.

4 Policy advice

In economic situations in which a "planner" may be identified (*e.g.*, a central bank), the establishment of a viability kernel can be used to select policies that keep the dynamic process x inside the closed constraint set K. Once the kernel is established, choosing a satisficing policy is a relatively simple procedure. Before we explain it, let us look briefly at the kinds of actions a central bank planner undertakes.

Routinely, at every time interval typical of the bank, the planner announces a cash interest rate. A *Taylor rule* or an optimising rule[11] might be used to determine the "new" interest rate. The latter usually equals the old interest rate *plus* or *minus* a fraction of a percentage point. The process leading to the rate increment determination is typically based on explicit or implicit optimisation of a loss function, which contains a significant number of calibrated or estimated parameters.

If $\mathcal{V}_\Psi(K \times [0, 0.07])$ denotes the viability kernel of our constraint set $K \times [0, 0.07]$ for a system with dynamics $\Psi(x, u)$, then the following "generic" policy rule can be formulated (see *regulation maps* in [2]):

$$\begin{cases} \qquad \forall x \in \mathcal{V}_\Psi(K \times [0, 0.07]) \text{ apply instrument } u \in W \\ \\ \text{where} \quad W \equiv \Big\{u \in U \,:\, \Psi(x, u) \text{ is a direction tangent or inward to } \mathcal{V}_\Psi(K \times [0, 0.07])\Big\}. \end{cases} \tag{17}$$

So, $W \subset U$ is a set of instruments available at x that keep the system evolution inside $\mathcal{V}_\Psi(K \times [0, 0.07])$.

For a given viability problem this rule will be decomposed into two *normative* directives: within the interior of the viability kernel $\mathcal{V}_\Psi(K \times [0, 0.07])$,

[11] See *e.g.*, [34].

every (admissible) control can be used[12]; on the boundary of the kernel $\mathsf{fr}\,\mathcal{V}_\Psi(K \times [0, 0.07])$, a specific instrument (path) must be followed[13].

The effect of the central bank's optimisation process is similar to the application of the satisficing policy: both maintain x in $K \times [0, 0.07]$. However, as explained later, fewer (subjective) parameters are needed to establish a viability kernel, $\mathcal{V}_\Psi(K \times [0, 0.07])$, than to compute a minimising solution to the bank loss function. Also, the "relaxed" approach advocated by (17) offers the planner a possibility to strive to achieve other goals (*e.g.*, political), which were not used for the specification of K. (Perhaps they were difficult to specify mathematically or they arose after the constraint set K had been established.) This is not the case of an optimal solution that remains optimal for the original problem formulation only.

When the model is subjected to shocks whose magnitude can be estimated (or their distribution is known), the viability kernel will have to be such that $x(t) + \mathsf{ball}(x(t), \varepsilon(t)) \in \mathcal{V}_\Psi(K \times [0, 0.07])$ where radius $\varepsilon(t)$ will depend on the shock[14]. Then, the above policy prescription can be followed.

Policies obtained through viability analysis are 'robust' (or precautionary or preventative) in that they are based on the economic system's inertia making them naturally forward looking. This is so because knowledge of the system's inertia enables detection and avoidance of regions where prevailing economic conditions (such as a large output gap or accelerating inflation) make system control difficult or impossible.

5 A method for the determination of viability kernels

In the spirit of [19], we would like to develop a means of approximating the viability kernel of a dynamic system with specific constraints. To this end, we have implemented an algorithm in MATLAB for computing such approximations. In short, our algorithm divides the problem into a discrete set of points, and then assesses whether, when starting from each point, the dynamic evolution of the system can be slowed to a (nearly) steady state without leaving the constraint set in finite time. A point of steady state is

[12] See [2] page 99 and [25] Section 3.4.1.3.

[13] Unless a steady state has been reached. Also, see *ibidem*.

[14] This radius might equal an expected shock magnitude. It may also equal the size of the shock that occurs "once in 100 years", *etc.*

defined to be a point at which the system dynamics are equal to zero; that is to say, the system is stationary/steady - the system will not have changed given some time interval[15]. Those points that can be brought close enough to such a state are included in the kernel by our algorithm, whilst those that are not are excluded[16].

Let δ denote the *discretisation* at which we have chosen to analyse the problem. By dividing $K \times [0,\ 0.07]$ along each of its four vertices into δ evenly-spaced points, and then combining these together, we obtain a finite version of our metastate space, $P_\delta \subset K \times [0,\ 0.07]$, containing δ^4 points. Solving $K \times [0,\ 0.07]$ over P_δ then[17] simply involves separating these points into two sets: the set of viable points in P_δ which we will denote by $\mathcal{V}_\Psi^\delta(K \times [0,\ 0.07])$ and the set of nonviable points, $P_\delta \setminus \mathcal{V}_\Psi^\delta(K \times [0,\ 0.07])$. As the points in $\mathcal{V}_\Psi^\delta(K \times [0,\ 0.07])$ are all viable, we then get the result that $\mathcal{V}_\Psi^\delta(K \times [0,\ 0.07]) \subset \mathcal{V}_\Psi(K \times [0,\ 0.07])$, where $\mathcal{V}_\Psi(K \times [0,\ 0.07])$ is the "true" viability kernel. It is thought that if δ is sufficiently large, then $\mathcal{V}_\Psi^\delta(K \times [0,\ 0.07])$ can be interpolated to give a fairly accurate picture of the true kernel, $\mathcal{V}_\Psi(K \times [0,\ 0.07])$.

As mentioned, we approximate $\mathcal{V}_\Psi^\delta(K \times [0,\ 0.07])$ by the set of points for which the dynamic evolution of the system can be slowed "sufficiently" using instruments in U. Let us call this set $\mathcal{S}_\Psi^\delta(K \times [0,\ 0.07], \epsilon)$, then

$$\mathcal{S}_\Psi^\delta(K \times [0,\ 0.07], \epsilon) \approx \mathcal{V}_\Psi^\delta(K \times [0,\ 0.07]) \subset \mathcal{V}_\Psi(K \times [0,\ 0.07]), \qquad (18)$$

where $\epsilon > 0$ (small) is the threshold for the Euclidean norm of the change in our system's uncontrolled variables (output gap, interest rate and exchange rate), $||\Psi(x, 0)||$, below which we will consider our system to be nearly stationary.

Our method for determining membership of $\mathcal{S}_\Psi^\delta(K \times [0,\ 0.07], \epsilon)$ is to examine each $p \in P_\delta$, and consider whether a first-order Euler approximation of our system, $\Psi_1(x, u)$ (discrete-time approximation) with an initial state of $x(0) = p$ can be brought to a (nearly) steady state within some finite

[15] A nearly steady state should then be one where movement over any reasonably long time interval is insignificant.

[16] More correctly then, the algorithm determines a discretised version of a *viability domain*. There may for instance be "orbits" that never arrive at a steady state, but which nonetheless always remain within the constraint set. These will not be detected by our algorithm; although we plan to produce a tool that will detect such orbits in the future.

[17] That is to say, approximating a solution to $K \times [0,\ 0.07]$ by considering a finite subset of points, P_δ.

number of steps by choosing the $u \in U$ at each step, t, that minimises $||\Psi_1(x(t+1), 0)||$, subject to the requirement that we don't leave $K \times [0,\ 0.07]$ in doing so. That is, for any $x(t) \in K \times [0,\ 0.07]$, our algorithm chooses:

$$\begin{array}{rcl} u & = & \arg\min ||\Psi_1(x(t) + \Psi_1(x(t), u), 0)|| \\ \text{s.t.,} & & u \in U \\ \text{and} & & i(t) + u \in [0,\ 0.07] \\ \text{and} & & x(t) + \Psi(x(t), u) \in K \times [0,\ 0.07] \end{array} \tag{19}$$

Thus, from each $x(0) = p \in P_\delta$, we iterate the system for as many steps, $x(t)$, as it takes either to violate one of the constraints, or for the Euclidean norm to fall below our predetermined threshold, $\epsilon > 0$.

The state that the system will be steered towards will be determined by the system's dynamics and will be dependent on the current system position. Exerting control as in (19) will thus be either "effective" or "ineffective", contingent on the particular point under consideration. If we can consistently decelerate the system from $x(0) = p$, and if we can do so fast enough, then the algorithm will be able to bring the system velocity to below ϵ, in which case $p \in \mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon)$[18]. Otherwise, the control will not be effective in slowing the system, in which case either the system will leave the constraint set, or it will loop (or "orbit") infinitely[19], meaning we cannot establish that $p \in \mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon)$.

Once we have established $\mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon)$, all that remains to be done is to satisfy ourselves that $\mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon) \approx \mathcal{V}^\delta_\Psi(K \times [0,\ 0.07])$. This is step is not (yet) undertaken by our algorithm; rather we have satisfied ourselves of this property by comparing our algorithm's computations of $\mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon)$ for our different problems with results obtained using different methods (*e.g.*, we have use results from [19] where a method based on [16] was applied to the same monetary economics problem).

In what follows then, we present some results from running the algorithm on three-dimensional and four-dimensional versions of our problem, and provide a comparison of these results with those obtained in [24] and [22].

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[18]In fact, we could say that if it takes n iterations to establish that $p \in \mathcal{S}^\delta_\Psi(K \times [0,\ 0.07], \epsilon)$, then $\{x(t)\}_{t=0}^n$ could all reasonably be supposed to be in $\mathcal{V}_\Psi(K \times [0,\ 0.07])$. Our algorithm as it stands does not however make use of this inference in approximating $\mathcal{V}_\Psi(K \times [0,\ 0.07])$.

[19]As the algorithm is not interested in the content of $\Psi(\cdot, \cdot)$, but simply attempts to solve (19), it is technically undecidable as to whether the algorithm will ever finish. For this reason, the MATLAB implementation gives up after some maximum number of loops.

6 Viability kernels

6.1 The closed economy – a brief assessment

Only parts of the viability kernel for the closed economy model[20] were computed and discussed in [24]. Here, we show the entire kernel in Figure 3 left panel as well as the *liquidity trap* corner in the right panel, analysed in detail in [24]. The kernel was computed using the algorithm described in Section 5.

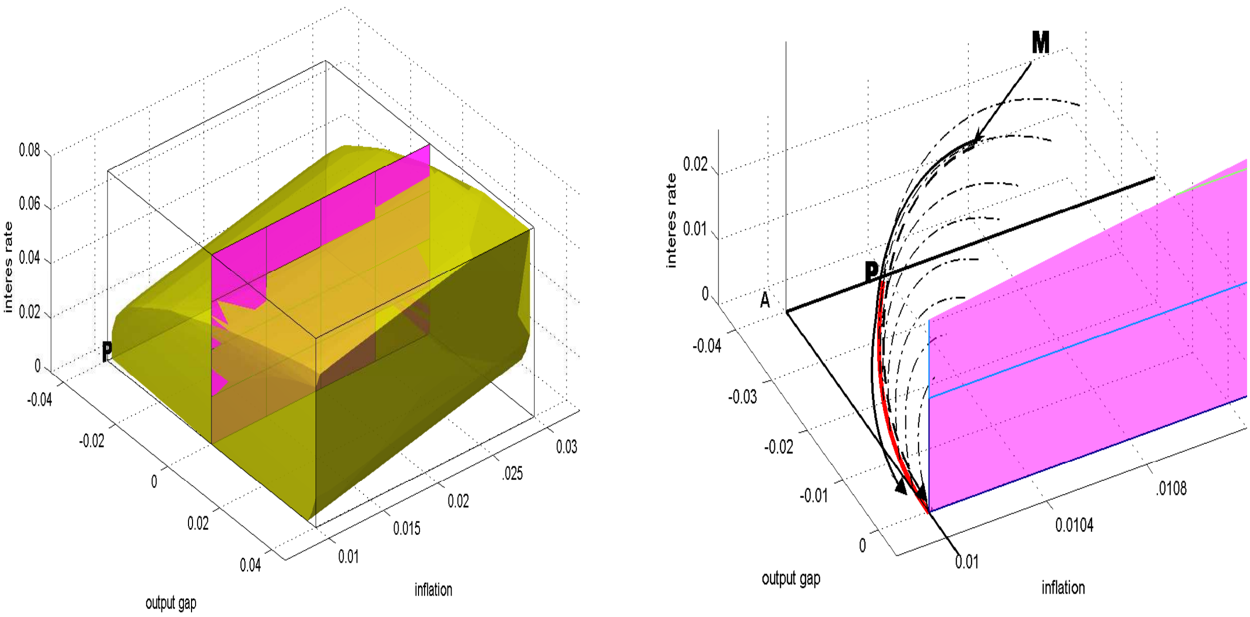

Figure 3: The closed economy kernel (right panel copied from [24]).

It should be rather evident that the panels present the same economy. However, they are drawn in different scales and using different methods. Nevertheless, we can identify the point **P** in both panels as the beginning of the zero-interest rate recovery evolution, from the maximum negative output

[20]This was a simpler model than (12)-(14) in that $\dot{q} \equiv 0$ and $q \equiv 0$. The model parameters define a generally "slower" dynamic system than the one analysed in this paper.

gap. Starting the evolution inside the kernel, point **M**, guarantees the system's obedience to the constraints, as per the dashed line in the right panel. On the other hand, a trajectory that does not belong to the kernel – the solid line in the right panel – ends up violating a constraint (here, the lower bound on inflation).

The advantage of an economic analysis using the "full" kernel is obvious. Facts that were explained in [24] with some effort by a series of graphs become evident when the analysis relies on the "full" kernel. For example, using the left panel, we can appreciate that entering the contractionary phase (negative output gap) should not happen when interest rate is high. (Almost) symmetrically, the interest rate cannot be low when the economy is booming *i.e.*, when output gap is positive and inflation is high.

6.2 A (small) open-economy monetary-policy analysis

We have applied the algorithm described in Section 5 to the four-dimensional system's dynamics (12)-(15). The figures presented below were obtained for 51 discretisation steps in each dimension.[21] We stress that all subsequent figures are *slices* taken from a 4D matrix, where the actual kernel "lives".

Figure 4 is obtained for $q = 0$ *i.e.*, when the local currency is in the neutral position. The slice resembles a cylinder centred around $y = 0$ and $\pi = i$. In broad terms, the larger the output gap, or the further the interest rate and inflation rate diverge from one another, the "less viable" the position becomes, when $q = 0$.

[21] So, the obtained kernels are results of an analysis of viability of 6.765201×10^6 points. The computation times on PCs or laptops are substantial and surpass 100 hours. However, sparser discretisations (like 10^4 or 21^4) generate (in minutes) similar, albeit coarser-edged, kernels.

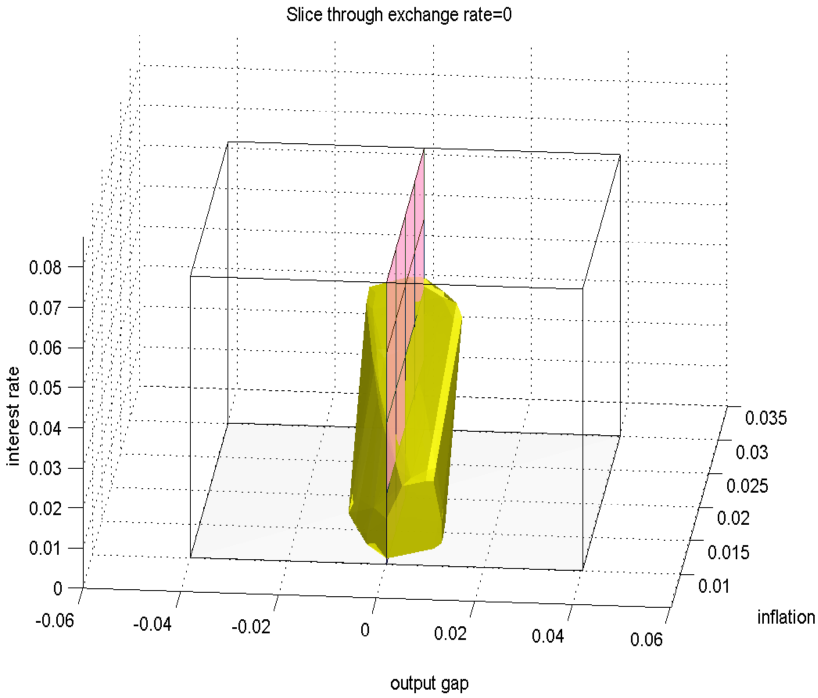

Figure 4: Kernel $\mathcal{V}_{\Psi}^{\delta}(K \times [0,\, 0.07])$ slice for $q = 0$.

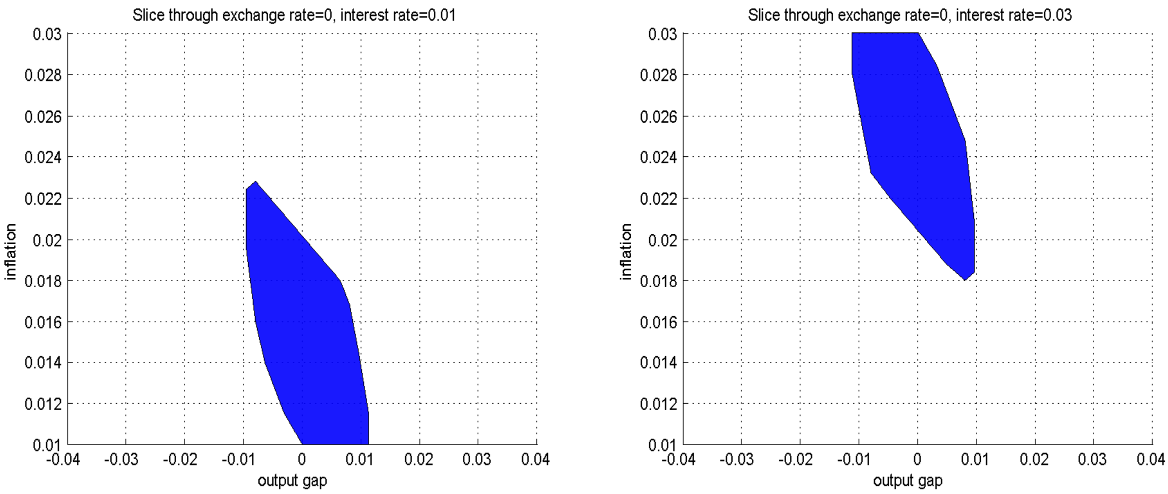

Figure 5: Kernel $\mathcal{V}_{\Psi}^{\delta}(K \times [0,\, 0.07])$ slices for $q = 0$ and $i = 1\%$; $i = 3\%$.

Interest rates in excess of 3% should be avoided, as the number of viable points quickly dwindles to zero beyond that level. Furthermore, a high interest rate can only be used in combination with either a high level of inflation or a mildly positive output gap (but not both together); a low interest rate correspondingly can only be used with a low level of inflation or a mildly negative output gap (but not both together). This is evident from Figure 5.

As judged by what we see in Figure 4 one could say that a "booming" economy (*i.e.*, when y is positive and large) is no more *viable* than a depressed economy. However, this statement needs be qualified.

Consider Figure 6 which presents two kernel slices for *overvalued* (local) currency.

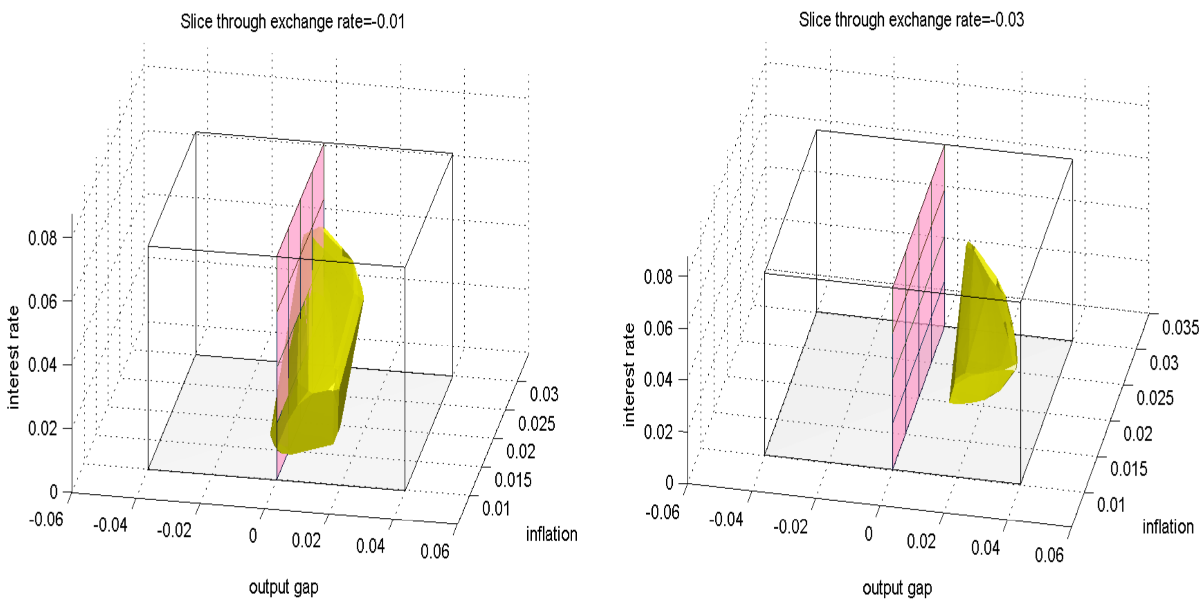

Figure 6: Kernel $\mathcal{V}_{\Psi}^{\delta}(K \times [0, 0.07])$ 3D slices for $q = -1\%$ and $q = -3\%$.

We can see that for mildly overvalued currency (left panel) the kernel moderately diminishes in "volume" relative to when the currency is in the neutral position (Figure 4), and that it has "shifted" somewhat to the right. The kernel continues to shrink and shifts further to the right as the currency becomes more overvalued (right panel). This rightward "shifting" indicates that a more overvalued currency requires a larger positive output gap, largely because the overvalued currency depresses local output, causing the output gap to diminish rapidly.

Further qualification of the statement that a hotter economy is no more *viable* than a less hot one is needed. Consider Figure 7, which presents two kernel slices for an *undervalued* currency. We can see that for mildly undervalued currency (left panel) the kernel shifts towards a negative output gap and moderately diminishes, relative to when the currency is in the neutral position (Figure 4). The kernel moves further to the left and shrinks further if the currency is more undervalued (right panel). We also notice that a depressed economy (*i.e.*, with negative outpur gap) copes well with an undervalued currency. Here, we have the reverse mechanism relative to that observed in Figure 6: low currency stimulates growth and the output gap swiftly increases.

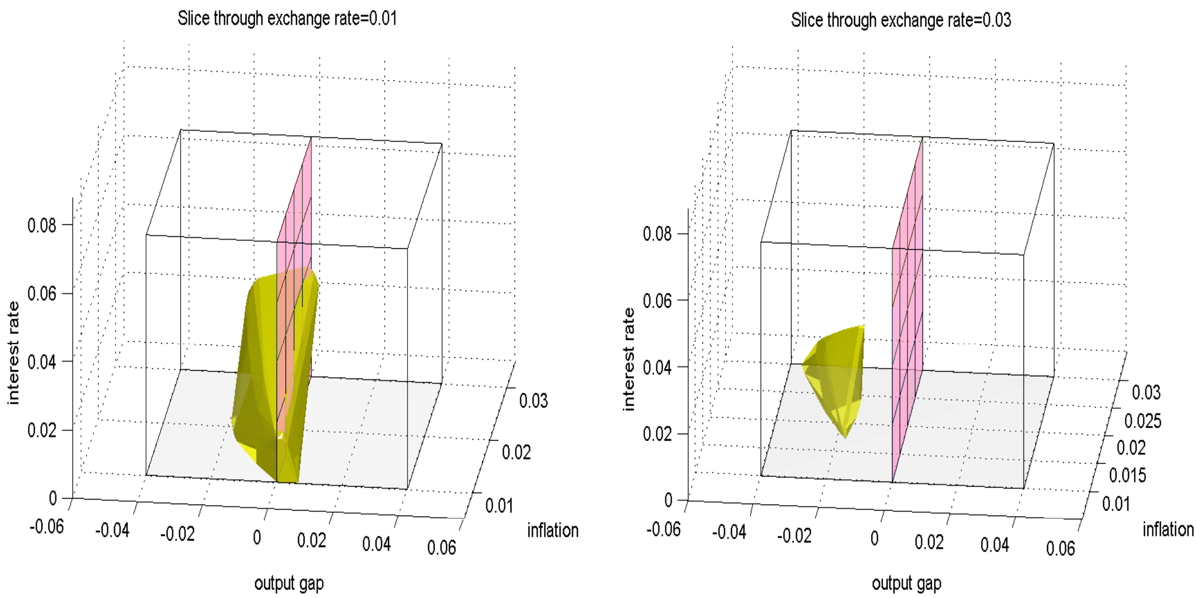

Figure 7: Kernel $\mathcal{V}_{\Psi}^{\delta}(K \times [0,\, 0.07])$ 3D slices for $q = 1\%$ and $i = 3\%$.

It is further worth noting that if q becomes larger than $\pm 4\%$, then the number of viable points quickly dwindles to zero, indicating that *only* mildly under- or over-valued currencies are manageable, given our restricted control.

These observations can be made more detailed if particular values of the interest rate and inflation are specified. For example, when the currency is mildly undervalued, high interest rates cannot be applied, and low inflation rates are not admissible. Somehow symmetrically, when the currency is mildly overvalued, low interest rates cannot be applied, and high inflation rates are not admissible.

Policy advice. Presenting the Central Bank governor with a viability kernel like $\mathcal{V}^{\delta}_{\Psi}(K\times[0,\ 0.07])$ helps them identify the critical states of the economy that should be avoided. On the other hand, the information that the *current* state of the economy is "well" inside $\mathcal{V}^{\delta}_{\Psi}(K\times[0,\ 0.07])$ is reassuring for the market and enables the governor (or government) to keep the economy in $K\times[0, 0.07]$ and, possibly, realise some other policy goals *i.e.*, beyond constraining the four variables that we have concerned ourselves with here.

We can clearly see that as the exchange rate rises, different parts of the economy become viable. In particular, a (slightly) depressed economy can cope with the mildly undervalued currency. On the other hand, to a mildly overvalued exchange rate ($q<0$), only parts of the booming economy are viable.

The fact that the viability kernels quickly diminish when the currency fluctuates confirm the conventional wisdom that managing a small-open economy is more difficult than a closed economy.

7 Concluding remarks

We propose that *viability theory* based on the concept of outcomes that are "good enough" provides a good environment for the monetary policy decision-making process, which is perhaps better than the frequently-used linear-quadratic optimisation framework. The advantages include an analysis of a *variety* of goals and the possibility of their loose (rather than strict) definition–either separate or joint. In short, Herbert Simon's contention that satisficing rather than optimising solutions capture the essence of desirable economic outcomes, can be assessed, and tentatively confirmed through this approach. Furthermore, economic results obtained through viability theory might be more robust to the actions of real economic agents.

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